Appendix to Dynamic Scoring: Some Lessons from the Neoclassical Growth Model

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1 Result (3)

The Ramsey model's Euler equation can be derived as follows. This derivation follows Barro & Sala-i-Martin (1999), chapter 2. For a derivation that avoids dynamic optimization, we recommend Romer (2001), chapter 2.

Assume there are H households of size $\frac{N}{H}$, each of which is infinitely-lived and representative of the economy. In this paper, we will assume population growth is zero to simplify the analysis. Each household derives utility from consumption $C_t = c_t e^{gt}$ per member, according to an instantaneous utility function $u(\cdot)$, that we assume to be CRRA. The discount rate is ρ. The present value of lifetime utility at time 0 is thus:

$$\begin{aligned} Utility &= \int_{t=0}^{\infty} e^{-\rho t} u(C_t) dt \\ \text{applying CRRA utility,} &= \int_{t=0}^{\infty} e^{-\rho t} \frac{(C_t)^{1-\gamma}}{1-\gamma} dt \end{aligned}$$

Each household has income from its labor input and from its initial capital holdings, since the households are assumed to own the capital of the economy, and households recieve their share T of transfers from the government. The level of the household's labor input is normalized to one and its per capita capital stock is $K = ke^{gt}$. The household receives wage w per unit of labor and a constant rate of return r on capital, but pays taxes of τ_k on capital income and τ_n on labor income. We can write the dynamic budget constraint of the household as

$$\dot{K} = (1-\tau_n)w + (1-\tau_k)rK - C - gK + T$$

To allow temporary indebtedness, the credit markets will require that the present value of household assets must be non-negative. That is:

$$\lim_{t\to\infty} K(t)e^{-rt} \geq 0$$

Maximization of utility subject to this budget constraint is a dynamic optimization problem. We set up a present-value Hamiltonian function and find first order conditions. After substitutions and simplifications, we derive equation (3), the so-called Euler equation, mentioned in the main text, which uses the after-tax rate of return $\tilde{r} = (1-\tau_k)r$.

$$
\begin{aligned}
H &= e^{-\rho t}\frac{(C_t)^{1-\gamma}}{1-\gamma} + \varphi(t)\left[(1-\tau_n)w + (1-\tau_k)rK - C + T\right] \\
FOC_K &: (1-\tau_k)r\varphi(t) = -\dot{\varphi}(t) \\
FOC_C &: e^{-\rho t}C_t^{-\gamma} = \varphi(t) \\
\frac{dFOC_C}{dt} &: \dot{\varphi}(t) = -\rho e^{-\rho t}C_t^{-\gamma} - \gamma e^{-\rho t}C_t^{-\gamma-1}\dot{C}_t \\
-(1-\tau_k)re^{-\rho t}C_t^{-\gamma} &= -\rho e^{-\rho t}C_t^{-\gamma} - \gamma e^{-\rho t}C_t^{-\gamma-1}\dot{C}_t \\
-(1-\tau_k)r &= -\rho - \gamma\frac{\dot{C}_t}{C_t} \\
(1-\tau_k)r &= \gamma\frac{\dot{c}(t)}{c(t)} + \rho + \gamma g \\
\tilde{r} &= \gamma\frac{\dot{c}}{c} + \rho + \gamma g
\end{aligned}
$$

In the steady state, this simplifies to $\tilde{r} = \rho + \gamma g$ (3)

which is result (3).

2 Results (5) and (6)

Here we derive the results for a change in either tax rate.

To proceed with our analysis, we totally differentiate (4) to obtain, suppressing arguments:

$$dR = kf'd\tau_k + \tau_k\left(kf'' + f'\right)dk + (f - kf')d\tau_n + \tau_n(-kf'')dk$$

We can use this result to determine the effect of a change in capital income tax τ_k or in labor income tax τ_n on total revenue.

First, divide by $d\tau_k$, and recognize that $\frac{d\tau_n}{d\tau_k} = 0$,since both are exogenously set, to get:

$$\frac{dR}{d\tau_k} = kf' + \left[\tau_k\left(kf'' + f'\right) + \tau_n(-kf'')\right]\frac{dk}{d\tau_k}$$

Performing some algebra, we can put this into a more easily interpreted form.

$$
\begin{aligned}
\frac{dR}{d\tau_k} &= kf' + \frac{\left[\tau_k(kf'' + f') - \tau_n(kf'')\right]f'}{(1-\tau_k)f''} \\
\frac{dR}{d\tau_k} &= kf'\left(1 + \frac{(\tau_k - \tau_n)f''}{(1-\tau_k)f''}\right) + \frac{\tau_k\left(f'\right)^2}{(1-\tau_k)f''} \\
\frac{dR}{d\tau_k} &= kf'\frac{(1-\tau_n)}{(1-\tau_k)} + \frac{\tau_k}{(1-\tau_k)}\frac{\left(f'\right)^2}{f''}
\end{aligned}
$$

Assuming Cobb-Douglas, we can simplify this result to obtain result (5):

$$\begin{aligned}
\frac{dR}{d\tau_k} &= \left[\frac{(1-\tau_n)}{(1-\tau_k)} + \frac{\tau_k}{(1-\tau_k)(\alpha-1)}\right]\alpha f(k). \\
\frac{dR}{d\tau_k} &= \left[\frac{(1-\tau_n)(\alpha-1)+\tau_k}{(1-\tau_k)(\alpha-1)}\right]\alpha f(k). \\
\frac{dR}{d\tau_k} &= \left[\frac{\alpha-1-\tau_n\alpha+\tau_n+\tau_k}{(1-\tau_k)(\alpha-1)}\right]\alpha f(k). \\
\frac{dR}{d\tau_k} &= \left[\frac{(1-\tau_k)(\alpha-1)-\tau_n\alpha+\tau_n+\tau_k+\tau_k(\alpha-1)}{(1-\tau_k)(\alpha-1)}\right]\alpha f(k). \\
\frac{dR}{d\tau_k} &= \left[1-\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\right]\alpha f(k) \qquad (5)
\end{aligned}$$

Next, we find the analogous expression for a change in the tax on labor income, τ_n. Recall that $\frac{d\tau_k}{d\tau_n}=0$,since both are exogenously set. Use equations (1) and (3) to note that

$$\frac{dk}{d\tau_n}=0$$

Then,

$$\begin{aligned}
\frac{dR}{d\tau_n} &= (f-kf') + \left[\tau_k\left(kf''+f'\right)+\tau_n(-kf'')\right]\frac{dk}{d\tau_n} \\
\frac{dR}{d\tau_n} &= (f-kf') \qquad (6)
\end{aligned}$$

If we assume Cobb-Douglas, this simplifies to result (6):

$$\frac{dR}{d\tau_n}=(1-\alpha)k^\alpha$$

3 Results (10)-(11)

To derive the Euler equation of our more general model, we again solve a dynamic optimization problem, using expressions for utility and the dynamic budget constraint found in the paper's main text.

$$\begin{aligned}
H &= \left[\begin{array}{c} e_t^{-\rho t}\frac{\left(c_t e^{gt}\right)^{1-\gamma}e^{(1-\gamma)v(N)}-1}{1-\gamma} + \\ \varphi(t)\left[(1-\tau_n)wN+(1-\tau_k)rk-c-gk+T\right] \end{array}\right] \\
FOC_k &: \left((1-\tau_k)r-g\right)\varphi = -\dot{\varphi} \\
FOC_c &: e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c(t)^{-\gamma} = \varphi \\
FOC_N &: e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N) = -(1-\tau_n)w(t)\varphi(t)
\end{aligned}$$

$$\frac{dFOC_c}{dt} \quad : \quad \dot{\varphi} = \begin{bmatrix} -\rho e^{-\rho t} e^{gt(1-\gamma)} e^{(1-\gamma)v(N)} c^{-\gamma} + \\ (1-\gamma) g e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma} + \\ v'(N)\dot{N} e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma} - \\ \gamma e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma-1} \dot{c} \end{bmatrix}$$

$$\begin{aligned}
-((1-\tau_k)r - g) e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma} &= \begin{bmatrix} -\rho e^{-\rho t} e^{gt(1-\gamma)} e^{(1-\gamma)v(N)} c^{-\gamma} + \\ (1-\gamma) g e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma} + \\ (1-\gamma) v'(N)\dot{N} e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma} - \\ \gamma e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma-1} \dot{c} \end{bmatrix} \\
-((1-\tau_k)r - g) &= -\rho + (1-\gamma)g + (1-\gamma)v'(N)\dot{N} - \gamma c^{-1}\dot{c} \\
-(1-\tau_k)r + g &= -\rho + (1-\gamma)g + (1-\gamma)v'(N)N\frac{\dot{N}}{N} - \gamma\frac{\dot{c}}{c} \\
(1-\tau_k)r &= \rho + \gamma g + \gamma\frac{\dot{c}}{c} + (1-\gamma)v'(N)N\frac{\dot{N}}{N} \\
\tilde{r} &= \rho + \gamma g \qquad (11)
\end{aligned}$$

The final step recognizes that $\frac{\dot{c}}{c}$ and $\frac{\dot{N}}{N}$ will be zero in the steady state, as consumption per efficiency unit (we assume no population growth for simplicity) and labor supply are constant in the steady state. This is result (11).

Combining FOC_c with FOC_n, we derive result (10):

$$\begin{aligned}
\frac{e^{-\rho t} e^{(1-\gamma)gt} e^{(1-\gamma)v(N)} c^{-\gamma}}{e^{-\rho t} e^{gt(1-\gamma)} c(t)^{1-\gamma} e^{(1-\gamma)v(N)} v'(N)} &= \frac{\varphi}{-(1-\tau_n)w\varphi} \\
v'(N) &= \frac{-(1-\tau_n)w}{c} \qquad (10)
\end{aligned}$$

4 Result (12)

To derive (12), we note the dynamic budget constraint.

$$\dot{k} = (1-\tau_n)wN + (1-\tau_k)rk - c - gk + T$$

In the steady state, $\dot{k}$ is equal to zero. We know that T is equal to the sum of tax revenue: $T = \tau_n wN + \tau_k rk$. Thus, we can rewrite the budget constraint as

$$c = f(k, N) - gk$$

This is result (12) in the main text.

5 Results (14-15), General Production Technology

Now we analyze the main model with non-Cobb-Douglas production. Again, some of the steady-state equations change:

$$f(k, n) = f(k, n). \qquad (7)$$

$$r = f_k. \tag{8}$$

$$w = f_n. \tag{9}$$

$$v'(n) = \frac{-\left(1-\tau_n\right)w}{c}. \tag{10}$$

$$r = \frac{\rho + \gamma g}{1-\tau_k}. \tag{11}$$

$$c = f - gk \tag{12}$$

$$R = \tau_k rk + \tau_n wn. \tag{13}$$

From CRS, we know that

$$\begin{aligned} \alpha &= \frac{kf_k}{f} && \text{(A11)} \\ (1-\alpha) &= \frac{f-kf_k}{f} = \frac{nf_n}{f} && (1) \end{aligned}$$

But note that α is no longer fixed and, thus, cannot be treated as a parameter in our derivations. The elasticity of substitution is (as in Hicks 1932)

$$\xi = \frac{f_n f_k}{f f_{kn}}$$

For future reference, we collect expressions:

$$\begin{aligned} \xi &= \frac{f_n f_k}{f f_{kn}} && \text{(A11)} \\ \alpha &= \frac{kf_k}{f} && (2) \\ 1-\alpha &= 1-\frac{kf_k}{f} = \frac{f-kf_k}{f} = \frac{nf_n}{f} && (3) \end{aligned}$$

We also know that

$$\begin{aligned} f_{kk} &= -f_{kn}\frac{n}{k} \\ f_{nn} &= -f_{nk}\frac{k}{n} \\ f_{kn} &= f_{nk} \end{aligned}$$

5.0.1 Capital Tax Cut

To derive these results, we use the system of two equations that simplifies (7-12):

$$(1-\tau_k)f_k - (\rho + \gamma g) = 0.$$

$$v'(N)\cdot(f-gk) + \left(1-\tau_n\right)f_n = 0.$$

From the first, we can write

$$f_k = \frac{\rho + \gamma g}{1 - \tau_k}$$

For $\frac{dk}{d\tau_k}$, take the total derivative of the first of these:

$$d\tau_k\left[-f_k\right] + d\tau_n\left[0\right] + dN\left[(1-\tau_k)f_{kN}\right] + dk\left[(1-\tau_k)f_{kk}\right] = 0$$

$$\frac{dk}{d\tau_k} = \frac{f_k - (1-\tau_k)f_{kN}\frac{dN}{d\tau_k}}{(1-\tau_k)f_{kk}}.$$

For $\frac{dN}{d\tau_k}$, again apply the implicit function theorem, this time to the second of these.

$$v'(N)\cdot(f-gk) + (1-\tau_n)\,f_n = 0.$$

$$Nv'(N) + \frac{(1-\tau_n)Nf_n}{(f-gk)} = 0$$

Take the total derivative of this,

$$\left\{\begin{array}{c} dN\left[Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2}\right] \\ +dk\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \\ +d\tau_n\left[\frac{-Nf_n}{f-gk}\right] + d\tau_k\left[0\right] \end{array}\right\} = 0.$$

Dividing by $d\tau_k$,

$$\left\{\begin{array}{c} \frac{dN}{d\tau_k}\left[Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2}\right] \\ = -\frac{dk}{d\tau_k}\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \end{array}\right\}$$

which combines with:

$$\frac{dk}{d\tau_k} = \frac{f_k - (1-\tau_k)f_{kN}\frac{dN}{d\tau_k}}{(1-\tau_k)f_{kk}}.$$

to give:

$$\left\{\begin{array}{c} \frac{dN}{d\tau_k}\left[Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2}\right] \\ = -\frac{f_k-(1-\tau_k)f_{kN}\frac{dN}{d\tau_k}}{(1-\tau_k)f_{kk}}\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \end{array}\right\}$$

Collecting terms

$$\left\{\begin{array}{c} \frac{dN}{d\tau_k}\left[\begin{array}{c} Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \\ -\frac{(1-\tau_k)f_{kN}}{(1-\tau_k)f_{kk}}\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2} \end{array}\right] \\ = -\frac{f_k}{(1-\tau_k)f_{kk}}\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \end{array}\right\}$$

Use

$$f_{kk} = -f_{kn}\frac{n}{k}$$

and

$$\xi = \frac{f_n f_k}{f f_{kn}} = -\frac{N f_n f_k}{f f_{kk} k} = -(1-\alpha)\frac{f_k}{k f_{kk}}$$

To rewrite this as

$$\left\{ \begin{array}{c} \frac{dN}{d\tau_k}\left[\begin{array}{c} Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_n f_n}{(f-gk)^2} \\ +\frac{k(f-gk)(1-\tau_n)f_{nk}-k(1-\tau_n)f_n(f_k-g)}{(f-gk)^2} \end{array} \right] \\ = \frac{k}{(1-\tau_k)}\frac{\xi}{(1-\alpha)}\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \end{array} \right\}$$

Simplify

$$\left\{ \begin{array}{c} \frac{dN}{d\tau_k}\left[\begin{array}{c} Nv''(N) + v'(N) \\ +\frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n+kf_{nk})-(1-\tau_n)f_n(Nf_n+k(f_k-g))}{(f-gk)^2} \end{array} \right] \\ = \frac{k}{(1-\tau_k)}\frac{\xi}{(1-\alpha)}\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \end{array} \right\}$$

Note that $Nf_{nn} + kf_{nk} = 0$, so

$$\left\{ \begin{array}{c} \frac{dN}{d\tau_k}\left[\begin{array}{c} Nv''(N) + v'(N) \\ +\frac{(f-gk)(1-\tau_n)f_n-(1-\tau_n)f_n(Nf_n+k(f_k-g))}{(f-gk)^2} \end{array} \right] \\ = \frac{k}{(1-\tau_k)}\frac{\xi}{(1-\alpha)}\left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2}\right] \end{array} \right\}$$

Combining terms,

$$\left\{ \begin{array}{c} \frac{dN}{d\tau_k}\left[\begin{array}{c} Nv''(N) + v'(N) \\ +\frac{(1-\tau_n)f_n[(f-gk)-Nf_n-k(f_k-g)]}{(f-gk)^2} \end{array} \right] \\ = \frac{k}{(1-\tau_k)}\frac{\xi}{(1-\alpha)}\left[\frac{(1-\tau_n)N((f-gk)f_{nk}-f_n(f_k-g))}{(f-gk)^2}\right] \end{array} \right\}$$

Simplify

$$\left\{ \begin{array}{c} \frac{dN}{d\tau_k}\left[\begin{array}{c} Nv''(N) + v'(N) \\ +\frac{(1-\tau_n)f_n[f-Nf_n-kf_k]}{(f-gk)^2} \end{array} \right] \\ = \frac{k}{(1-\tau_k)}\frac{\xi}{(1-\alpha)}\left[\frac{(1-\tau_n)N((f-gk)f_{nk}-f_n(f_k-g))}{(f-gk)^2}\right] \end{array} \right\}$$

Note that $[f - Nf_n - kf_k] = 0$, so this becomes

$$\left\{ \begin{array}{c} \frac{dN}{d\tau_k}[Nv''(N) + v'(N)] \\ = \frac{k}{(1-\tau_k)}\frac{\xi}{(1-\alpha)}\left[\frac{(1-\tau_n)N((f-gk)f_{nk}-f_n(f_k-g))}{(f-gk)^2}\right] \end{array} \right\}$$

or, rearranging,

$$\left\{ \frac{dN}{d\tau_k} = \frac{(1-\tau_n)}{(1-\tau_k)}\frac{\xi k N}{(1-\alpha)}\left[\frac{f_{nk}}{(f-gk)} - \frac{f_n(f_k-g)}{(f-gk)^2}\right]\frac{1}{[Nv''(N)+v'(N)]} \right\}$$

From our results for the elasticity of labor supply, we know that

$$Nv''(N) + v'(N) = v'(N)N\left(\frac{1+\sigma}{N\sigma}\right).$$

thus, using our result from above that

$$Nv'(N) + \frac{(1-\tau_n)Nf_n}{(f-gk)} = 0$$

we know that

$$Nv''(N) + v'(N) = \frac{-\left(1-\tau_n\right)Nf_n}{(f-gk)}\left(\frac{1+\sigma}{N\sigma}\right)$$

Plugging in this expression,

$$\left\{\frac{dN}{d\tau_k} = \frac{(1-\tau_n)}{(1-\tau_k)}\frac{\xi kN}{(1-\alpha)}\left[\frac{f_{nk}}{(f-gk)} - \frac{f_n\left(f_k-g\right)}{(f-gk)^2}\right]\frac{(f-gk)}{-\left(1-\tau_n\right)Nf_n}\left(\frac{N\sigma}{1+\sigma}\right)\right\}$$

Simplifying,

$$\left\{\frac{dN}{d\tau_k} = -\frac{1}{(1-\tau_k)}\frac{\xi kN}{(1-\alpha)}\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\left(\frac{\sigma}{1+\sigma}\right)\right\}$$

or,

$$\left\{\frac{dN}{d\tau_k} = \frac{-1}{(1-\tau_k)(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\left(\frac{N\sigma}{1+\sigma}\right)\right\}$$

We use this in our expression for $\frac{dk}{d\tau_k}$

$$\frac{dk}{d\tau_k} = \frac{f_k - (1-\tau_k)f_{kN}\frac{dN}{d\tau_k}}{(1-\tau_k)f_{kk}}.$$

to obtain

$$\frac{dk}{d\tau_k} = \frac{f_k - (1-\tau_k)f_{kN}\frac{-1}{(1-\tau_k)(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\left(\frac{N\sigma}{1+\sigma}\right)}{(1-\tau_k)f_{kk}}.$$

Simplifying,

$$\frac{dk}{d\tau_k} = \frac{f_k + \frac{f_{kN}}{(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\left(\frac{N\sigma}{1+\sigma}\right)}{(1-\tau_k)f_{kk}}.$$

Separating terms, this is

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k)f_{kk}} + \frac{\frac{f_{kN}}{(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]}{(1-\tau_k)f_{kk}}\frac{N\sigma}{1+\sigma}.$$

Use

$$\xi = \frac{f_n f_k}{f f_{kn}} = -\frac{N f_n f_k}{f f_{kk} k} = -(1-\alpha)\frac{f_k}{k f_{kk}}$$

To simplify this expression

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k) f_{kk}} + \frac{-\frac{f_k N}{(1-\alpha)}(1-\alpha)\frac{f_k}{f_{kk}}\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right]}{(1-\tau_k) f_{kk}} \frac{N\sigma}{1+\sigma}.$$

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k) f_{kk}} + \frac{-f_k N\left(\frac{f_k}{k f_{kk}}\right) k\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right]}{(1-\tau_k) f_{kk}} \frac{N\sigma}{1+\sigma}.$$

Then use

$$f_{kk} = -f_{kn}\frac{n}{k}$$

to derive

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k) f_{kk}} + \frac{f_k k\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right]}{(1-\tau_k) f_{kk}} \frac{\sigma}{1+\sigma}.$$

Rearranging,

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k) f_{kk}} + \frac{f_k}{(1-\tau_k) f_{kk}} k\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right] \frac{\sigma}{1+\sigma}.$$

Simplifying,

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k) f_{kk}}\left[1 + k\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right] \frac{\sigma}{1+\sigma}\right].$$

Now, we need an expression for $\frac{dR}{d\tau_k}$

$$\begin{aligned}
R &= \tau_k r k + \tau_n w N \\
R &= \tau_k f_k k + \tau_n f_n N \\
\left.\frac{dR}{d\tau_k}\right|_{dynamic} &= \left\{\begin{array}{c} f_k k \\ +\frac{dk}{d\tau_k}\left[\tau_k (f_{kk} k + f_k) + \tau_n f_{nk} N\right] \\ +\frac{dN}{d\tau_k}\left[\tau_k f_{kn} k + \tau_n (f_{nn} N + f_n)\right] \end{array}\right\}
\end{aligned}$$

From CRS, we know that:

$$\begin{aligned}
f_{kk} &= -f_{kn}\frac{n}{k} \\
f_{nn} &= -f_{nk}\frac{k}{n} \\
f_{kn} &= f_{nk}
\end{aligned}$$

So this expression becomes

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{\begin{array}{c} f_k k \\ +\frac{dk}{d\tau_k}\left[\tau_k\left(-f_{kn} N + f_k\right) + \tau_n f_{nk} N\right] \\ +\frac{dN}{d\tau_k}\left[\tau_k f_{kn} k + \tau_n\left(-f_{nk} k + f_n\right)\right] \end{array}\right\}$$

Rearranging,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{dk}{d\tau_k} f_k \tau_k + \frac{dN}{d\tau_k} f_n \tau_n \\ +\frac{dk}{d\tau_k} N \left[(\tau_n - \tau_k) f_{nk} \right] \\ +\frac{dN}{d\tau_k} k \left[(\tau_k - \tau_n) f_{kn} \right] \end{array} \right\}$$

Or simply

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{dk}{d\tau_k} f_k \tau_k + \frac{dN}{d\tau_k} f_n \tau_n \\ +\left(\frac{dk}{d\tau_k} N - \frac{dN}{d\tau_k} k \right) \left[(\tau_n - \tau_k) f_{nk} \right] \end{array} \right\}$$

Now, using our expressions for $\frac{dk}{d\tau_k}$ and $\frac{dN}{d\tau_k}$,

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k) f_{kk}} \left[1 + k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right].$$

$$\left\{ \frac{dN}{d\tau_k} = \frac{-1}{(1-\tau_k)(1-\alpha)} \xi k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \left(\frac{N\sigma}{1+\sigma} \right) \right\}$$

we can show:.

$$\left\{ \begin{array}{c} \left(\frac{dk}{d\tau_k} N - \frac{dN}{d\tau_k} k \right) = \\ \left\{ \frac{f_k}{(1-\tau_k) f_{kk}} \left[1 + k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right] \right\} N \\ -\left\{ \frac{-1}{(1-\tau_k)(1-\alpha)} \xi k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \left(\frac{N\sigma}{1+\sigma} \right) \right\} k \end{array} \right\}$$

simplifies to:

$$\left\{ \begin{array}{c} \left(\frac{dk}{d\tau_k} N - \frac{dN}{d\tau_k} k \right) = \\ \left\{ \frac{f_k}{(1-\tau_k) f_{kk}} \left[1 + k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right] \right\} N \\ -\left\{ \frac{f_k}{(1-\tau_k) f_{kk}} \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{N\sigma}{1+\sigma} \right\} k \end{array} \right\}$$

thus,

$$\left\{ \begin{array}{c} \left(\frac{dk}{d\tau_k} N - \frac{dN}{d\tau_k} k \right) = \\ \left\{ \frac{f_k}{(1-\tau_k) f_{kk}} \left[N + k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{N\sigma}{1+\sigma} \right] \right\} \\ -\left\{ \frac{f_k}{(1-\tau_k) f_{kk}} \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{N\sigma}{1+\sigma} \right\} k \end{array} \right\}$$

Rearranging,

$$\left\{ \begin{array}{c} \left(\frac{dk}{d\tau_k} N - \frac{dN}{d\tau_k} k \right) = \\ \frac{f_k}{(1-\tau_k) f_{kk}} N \\ +\frac{f_k}{(1-\tau_k) f_{kk}} k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{N\sigma}{1+\sigma} \\ -\frac{f_k}{(1-\tau_k) f_{kk}} k \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{N\sigma}{1+\sigma} \end{array} \right\}$$

Or, cancelling terms,

$$\left(\frac{dk}{d\tau_k}N - \frac{dN}{d\tau_k}k\right) = \frac{f_k}{(1-\tau_k)f_{kk}}N$$

So,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{dk}{d\tau_k} f_k \tau_k + \frac{dN}{d\tau_k} f_n \tau_n \\ + \left(\frac{dk}{d\tau_k}N - \frac{dN}{d\tau_k}k\right) \left[(\tau_n - \tau_k) f_{nk}\right] \end{array} \right\}$$

becomes

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{dk}{d\tau_k} f_k \tau_k + \frac{dN}{d\tau_k} f_n \tau_n \\ + \frac{f_k}{(1-\tau_k)f_{kk}} N \left[(\tau_n - \tau_k) f_{nk}\right] \end{array} \right\}$$

or, rearranging,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{dk}{d\tau_k} f_k \tau_k + \frac{dN}{d\tau_k} f_n \tau_n \\ + \frac{1}{(1-\tau_k)} \frac{n f_k f_{nk}}{f_{kk}} \left[(\tau_n - \tau_k)\right] \end{array} \right\}$$

using

$$f_{kk} = -f_{kn}\frac{n}{k}$$

this is,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{dk}{d\tau_k} f_k \tau_k + \frac{dN}{d\tau_k} f_n \tau_n \\ - \frac{k f_k}{(1-\tau_k)} \left[(\tau_n - \tau_k)\right] \end{array} \right\}$$

Then, substitute in our expressions for $\frac{dk}{d\tau_k}$ and $\frac{dN}{d\tau_k}$

$$\frac{dk}{d\tau_k} = \frac{f_k}{(1-\tau_k)f_{kk}}\left[1 + k\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right]\frac{\sigma}{1+\sigma}\right].$$

$$\frac{dN}{d\tau_k} = \frac{-1}{(1-\tau_k)(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)}\right]\left(\frac{N\sigma}{1+\sigma}\right)$$

to obtain

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k \\ + \left(\frac{f_k}{(1-\tau_k)f_{kk}}\left[1 + k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\frac{\sigma}{1+\sigma}\right]\right) f_k \tau_k \\ + \frac{-1}{(1-\tau_k)(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\left(\frac{N\sigma}{1+\sigma}\right) f_n \tau_n \\ - \frac{f_k}{(1-\tau_k)} k \left[(\tau_n - \tau_k)\right] \end{array} \right\}$$

Simplifying,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{f_k}{(1-\tau_k)f_{kk}} f_k \tau_k \\ + \left(\frac{f_k}{(1-\tau_k)f_{kk}}\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\frac{\sigma}{1+\sigma}\right) k f_k \tau_k \\ + \frac{-1}{(1-\tau_k)(1-\alpha)}\xi k\left[\frac{f_{nk}}{f_n} - \frac{(f_k-g)}{(f-gk)}\right]\left(\frac{\sigma}{1+\sigma}\right) N f_n \tau_n \\ - \frac{f_k}{(1-\tau_k)} k \left[(\tau_n - \tau_k)\right] \end{array} \right\}$$

or,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{f_k}{(1-\tau_k) f_{kk}} f_k \tau_k \\ + \left(\frac{f_k}{(1-\tau_k) f_{kk}} \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right) k f_k \tau_k \\ + \left(\frac{f_k}{(1-\tau_k) f_{kk}} \right) \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \left(\frac{\sigma}{1+\sigma} \right) N f_n \tau_n \\ - \frac{f_k}{(1-\tau_k)} k \left[(\tau_n - \tau_k) \right] \end{array} \right\}$$

and thus, collecting terms and using our expressions for α and $(1-\alpha)$,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{f_k}{(1-\tau_k) f_{kk}} f_k \tau_k \\ + \left(\frac{f_k}{(1-\tau_k) f_{kk}} \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right) \left(\alpha f \tau_k + (1-\alpha) f \tau_n \right) \\ - \frac{f_k}{(1-\tau_k)} k \left[(\tau_n - \tau_k) \right] \end{array} \right\}$$

rearranging,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} f_k k + \frac{f_k}{(1-\tau_k) f_{kk}} f_k \tau_k - \frac{f_k}{(1-\tau_k)} k \left[(\tau_n - \tau_k) \right] \\ + \left(\frac{f_k}{(1-\tau_k) f_{kk}} \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right) f \left(\alpha \tau_k + (1-\alpha) \tau_n \right) \end{array} \right\}$$

Pull out $f_k k$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 + \frac{f_k}{(1-\tau_k) k f_{kk}} \tau_k - \frac{(\tau_n - \tau_k)}{(1-\tau_k)} \\ + \left(\frac{f}{(1-\tau_k) k f_{kk}} \left[\frac{f_{nk}}{f_n} - \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right) \left(\alpha \tau_k + (1-\alpha) \tau_n \right) \end{array} \right\}$$

Now, recall that

$$\frac{-1}{(1-\alpha)} \xi = \frac{f_k}{k f_{kk}}$$

and simplify

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{1}{(1-\alpha)(1-\tau_k)} \xi \tau_k - \frac{(1-\alpha)(\tau_n - \tau_k)}{(1-\alpha)(1-\tau_k)} \\ + \left(\frac{1}{(1-\tau_k)} \left[\frac{f}{k f_{kk}} \frac{f_{nk}}{f_n} - \frac{f}{k f_{kk}} \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right) \left(\alpha \tau_k + (1-\alpha) \tau_n \right) \end{array} \right\}$$

And, using

$$f_{kk} = -f_{kn} \frac{n}{k}$$

derive

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{1}{(1-\alpha)(1-\tau_k)} \xi \tau_k - \frac{(1-\alpha)(\tau_n - \tau_k)}{(1-\alpha)(1-\tau_k)} \\ + \left(\frac{1}{(1-\tau_k)} \left[\frac{-f}{N} \frac{1}{f_n} - \frac{f}{k f_{kk}} \frac{f}{f} \frac{f_k}{f_k} \frac{(f_k - g)}{(f - gk)} \right] \frac{\sigma}{1+\sigma} \right) \left(\alpha \tau_k + (1-\alpha) \tau_n \right) \end{array} \right\}$$

simplifying,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{1}{(1-\alpha)(1-\tau_k)} \xi \tau_k - \frac{(1-\alpha)(\tau_n - \tau_k)}{(1-\alpha)(1-\tau_k)} \\ + \left(\frac{1}{(1-\tau_k)} \left[\frac{-1}{(1-\alpha)} - \frac{f_k}{k f_{kk}} \frac{f}{(f - gk)} \frac{(f_k - g)}{f_k} \right] \frac{\sigma}{1+\sigma} \right) \left(\alpha \tau_k + (1-\alpha) \tau_n \right) \end{array} \right\}$$

or

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{1}{(1-\alpha)(1-\tau_k)}\xi\tau_k - \frac{(1-\alpha)(\tau_n-\tau_k)}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{1}{(1-\tau_k)(1-\alpha)}\left[1 + (1-\alpha)\frac{f_k}{kf_{kk}}\frac{f}{(f-gk)}\frac{(f_k-g)}{f_k}\right]\frac{\sigma}{1+\sigma}\right)(\alpha\tau_k + (1-\alpha)\tau_n) \end{array} \right\}$$

Again, using

$$\frac{-1}{(1-\alpha)}\xi = \frac{f_k}{kf_{kk}}$$

derive

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{1}{(1-\alpha)(1-\tau_k)}\xi\tau_k - \frac{(1-\alpha)(\tau_n-\tau_k)}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{1}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{f}{f-gk}\frac{f_k-g}{f_k}\right]\frac{\sigma}{1+\sigma}\right)(\alpha\tau_k + (1-\alpha)\tau_n) \end{array} \right\}$$

Rearranging,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{f}{f-gk}\frac{f_k-g}{f_k}\right]\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

simplifying,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{1}{1-g\frac{k}{f}}\left(1 - \frac{g}{f_k}\right)\right]\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

or, using our expression for α,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{1-\frac{g}{f_k}}{1-\frac{\alpha g}{f_k}}\right]\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

simplifying

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{f_k-g}{f_k-\alpha g}\right]\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

We know that

$$f_k = \frac{\rho+\gamma g}{1-\tau_k}$$

So, this becomes

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{\frac{\rho+\gamma g}{1-\tau_k}-g}{\frac{\rho+\gamma g}{1-\tau_k}-\alpha g}\right]\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

or,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\left(\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)}\left[1 - \xi\frac{\rho+\gamma g-g(1-\tau_k)}{\rho+\gamma g-\alpha g(1-\tau_k)}\right]\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

Rearranging,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ + \left(\frac{\alpha\tau_k + (1-\alpha)\tau_n}{\alpha g(1-\tau_k) - (\rho+\gamma g)} \frac{(\rho+\gamma g)(1-\xi) + g(1-\tau_k)(\xi-\alpha)}{(1-\tau_k)(1-\alpha)} \frac{\sigma}{1+\sigma} \right) \end{array} \right\}$$

Rearranging,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ - \frac{\alpha\tau_k + (1-\alpha)\tau_n}{(\rho+\gamma g) - \alpha g(1-\tau_k)} \\ \cdot \frac{(\rho+\gamma g)(1-\xi) + (1-\tau_k)(\xi-\alpha) g}{(1-\alpha)(1-\tau_k)} \frac{\sigma}{1+\sigma} \end{array} \right\}$$

If $g = 0$, this is

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ - \left(\frac{\alpha\tau_k + (1-\alpha)\tau_n}{(1-\tau_k)(1-\alpha)} (1-\xi) \frac{\sigma}{1+\sigma} \right) \end{array} \right\}$$

Note that if we are in the Cobb-Douglas model, where $\xi = 1$, then,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{\alpha\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ + \left(\frac{\alpha\tau_k + (1-\alpha)\tau_n}{\alpha g(1-\tau_k) - (\rho+\gamma g)} \frac{\sigma}{1+\sigma} g \right) \end{array} \right\}$$

Our result from before.

5.0.2 Labor Tax Cut

To derive these results, we use the system of two equations that simplifies (7-12):

$$(1-\tau_k) f_k - (\rho + \gamma g) = 0.$$

$$v'(N) \cdot (f - gk) + (1-\tau_n) f_n = 0.$$

From the first, we can write

$$f_k = \frac{\rho + \gamma g}{1 - \tau_k}$$

For $\frac{dk}{d\tau_n}$, take the total derivative of the first of these:

$$d\tau_k [-f_k] + d\tau_n [0] + dN [(1-\tau_k) f_{kN}] + dk [(1-\tau_k) f_{kk}] = 0$$

which yields

$$\frac{dk}{d\tau_n} = -\frac{dN}{d\tau_n} \frac{f_{kN}}{f_{kk}}$$

or take the derivative of the whole thing with respect to τ_k:

$$\begin{aligned} (1-\tau_k) f_{kk} \frac{dk}{d\tau_n} + (1-\tau_k) f_{kn} \frac{dn}{d\tau_n} &= 0 \\ (1-\tau_k) f_{kk} \frac{dk}{d\tau_n} &= -(1-\tau_k) f_{kn} \frac{dn}{d\tau_n} \\ \frac{dk}{d\tau_n} &= -\frac{f_{kn}}{f_{kk}} \frac{dn}{d\tau_n} \end{aligned}$$

For $\frac{dN}{d\tau_n}$, again apply the implicit function theorem, this time to the second of these.

$$v'(N) \cdot (f - gk) + (1 - \tau_n)\, f_n = 0.$$

$$Nv'(N) + \frac{(1 - \tau_n)Nf_n}{(f - gk)} = 0$$

Take the total derivative of this,

$$\left\{ \begin{array}{c} dN \left[Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \right] \\ +dk \left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2} \right] \\ +d\tau_n \left[\frac{-Nf_n}{f-gk} \right] + d\tau_k\,[0] \end{array} \right\} = 0.$$

Dividing by $d\tau_n$,

$$\begin{array}{c} \frac{dN}{d\tau_n} \left[Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \right] \\ = -\frac{dk}{d\tau_n} \left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2} \right] + \frac{Nf_n}{f-gk} \end{array}$$

Combine with

$$\frac{dk}{d\tau_n} = -\frac{dN}{d\tau_n}\frac{f_{kN}}{f_{kk}}$$

to yield

$$\begin{array}{c} \frac{dN}{d\tau_n} \left[Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \right] \\ = \frac{dN}{d\tau_n}\frac{f_{kN}}{f_{kk}} \left[\frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2} \right] + \frac{Nf_n}{f-gk} \end{array}$$

Collecting terms,

$$\begin{array}{c} \frac{dN}{d\tau_n} \left[\begin{array}{c} Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \\ -\frac{f_{kN}}{f_{kk}} \frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2} \end{array} \right] \\ = \frac{Nf_n}{f-gk} \end{array}$$

Use

$$f_{kk} = -f_{kn}\frac{n}{k}$$

to rewrite this as

$$\begin{array}{c} \frac{dN}{d\tau_n} \left[\begin{array}{c} Nv''(N) + v'(N) + \frac{(f-gk)(1-\tau_n)(Nf_{nn}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \\ +\frac{k}{N} \frac{(f-gk)(1-\tau_n)Nf_{nk}-(1-\tau_n)Nf_n(f_k-g)}{(f-gk)^2} \end{array} \right] \\ = \frac{Nf_n}{f-gk} \end{array}$$

or, rearranging and using

$$f_{nn} = -f_{nk}\frac{k}{n}$$

we can derive,

$$\frac{dN}{d\tau_n}\left[\begin{array}{c} Nv''(N)+v'(N)+\frac{(f-gk)(1-\tau_n)(-kf_{nk}+f_n)-(1-\tau_n)Nf_nf_n}{(f-gk)^2} \\ +\frac{(f-gk)(1-\tau_n)kf_{nk}-(1-\tau_n)f_nk(f_k-g)}{(f-gk)^2} \end{array}\right] = \frac{Nf_n}{f-gk}$$

which simplifies to:

$$\frac{dN}{d\tau_n}\left[Nv''(N)+v'(N)+\frac{(f-gk)(1-\tau_n)(-kf_{nk}+f_n)-(1-\tau_n)Nf_nf_n+(f-gk)(1-\tau_n)kf_{nk}-(1-\tau_n)f_nk(f_k-g)}{(f-gk)^2}\right] = \frac{Nf_n}{f-gk}$$

Cancelling,

$$\frac{dN}{d\tau_n}\left[Nv''(N)+v'(N)+\frac{(f-gk)(1-\tau_n)(f_n)-(1-\tau_n)Nf_nf_n-(1-\tau_n)f_nk(f_k-g)}{(f-gk)^2}\right] = \frac{Nf_n}{f-gk}$$

which simplifies to

$$\frac{dN}{d\tau_n}\left[Nv''(N)+v'(N)+\frac{(1-\tau_n)f_n[(f-gk)-k(f_k-g)-Nf_n]}{(f-gk)^2}\right] = \frac{Nf_n}{f-gk}$$

Cancelling,

$$\frac{dN}{d\tau_n}\left[Nv''(N)+v'(N)+\frac{(1-\tau_n)f_n(f-kf_k-Nf_n)}{(f-gk)^2}\right] = \frac{Nf_n}{f-gk}$$

But we know that $(f-kf_k-Nf_n)=0$, so this becomes simply

$$\frac{dN}{d\tau_n} = \frac{Nf_n}{f-gk}\frac{1}{[Nv''(N)+v'(N)]}$$

From our results for the elasticity of labor supply, we know that

$$Nv''(N)+v'(N) = v'(N)N\left(\frac{1+\sigma}{N\sigma}\right).$$

thus, using our result from above that

$$Nv'(N)+\frac{(1-\tau_n)Nf_n}{(f-gk)} = 0$$

we know that

$$Nv''(N)+v'(N) = \frac{-(1-\tau_n)Nf_n}{(f-gk)}\left(\frac{1+\sigma}{N\sigma}\right)$$

So plugging this into our results, we obtain

$$\frac{dN}{d\tau_n} = \frac{Nf_n}{f-gk}\frac{(f-gk)}{-(1-\tau_n)Nf_n}\left(\frac{N\sigma}{1+\sigma}\right)$$

or simply

$$\frac{dN}{d\tau_n} = \frac{-N}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right)$$

Substituting this expression into that for $\frac{dk}{d\tau_n}$

$$\frac{dk}{d\tau_n} = -\frac{dN}{d\tau_n}\frac{f_{kN}}{f_{kk}}$$

gives

$$\frac{dk}{d\tau_n} = -\frac{-N}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right)\frac{f_{kN}}{f_{kk}}$$

or, using

$$f_{kk} = -f_{kn}\frac{n}{k}$$

this is:

$$\frac{dk}{d\tau_n} = \frac{-k}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right)$$

Now, we need an expression for $\frac{dR}{d\tau_n}$

$$\begin{aligned} R &= \tau_k rk + \tau_n wN \\ R &= \tau_k f_k k + \tau_n f_n N \end{aligned}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{ \begin{array}{c} f_n N + \\ \frac{dk}{d\tau_n}\left[\tau_k(f_{kk}k + f_k) + \tau_n f_{nk} N\right] \\ +\frac{dN}{d\tau_n}\left[\tau_k f_{kn} k + \tau_n(f_{nn}N + f_n\right] \end{array} \right\}$$

using

$$\begin{aligned} f_{kk} &= -f_{kn}\frac{n}{k} \\ f_{nn} &= -f_{nk}\frac{k}{n} \end{aligned}$$

obtain

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{ \begin{array}{c} f_n N + \\ \frac{dk}{d\tau_n}\left[\tau_k(-f_{kn}N + f_k) + \tau_n f_{nk} N\right] \\ +\frac{dN}{d\tau_n}\left[\tau_k f_{nk} k + \tau_n(-f_{nk}k + f_n\right] \end{array} \right\}$$

rearranging,

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{ \begin{array}{c} f_n N + \frac{dk}{d\tau_n}\tau_k f_k + \frac{dN}{d\tau_n}\tau_n f_n \\ \frac{dk}{d\tau_n}\left[\tau_k(-f_{kn}N) + \tau_n f_{nk} N\right] \\ +\frac{dN}{d\tau_n}\left[\tau_k f_{nk} k + \tau_n(-f_{nk}k\right] \end{array} \right\}$$

or simply

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{ \begin{array}{c} f_n N + \frac{dk}{d\tau_n}\tau_k f_k + \frac{dN}{d\tau_n}\tau_n f_n + \\ \left(\frac{dk}{d\tau_n}N - \frac{dN}{d\tau_n}k\right)\left[(\tau_n - \tau_k) f_{nk}\right] \end{array} \right\}$$

Now, we had from before the results that:

$$\frac{dN}{d\tau_n} = \frac{-N}{(1-\tau_n)}\frac{\sigma}{(1+\sigma)}.$$

$$\frac{dk}{d\tau_n} = \frac{-k}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right)$$

so, we can show

$$\left(\frac{dk}{d\tau_n}N - \frac{dN}{d\tau_n}k\right) = \left(\frac{-k}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right)N - \frac{-N}{(1-\tau_n)}\frac{\sigma}{(1+\sigma)}k\right)$$

or

$$\left(\frac{dk}{d\tau_n}N - \frac{dN}{d\tau_n}k\right) = \left(\frac{-kN}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right) + \frac{kN}{(1-\tau_n)}\frac{\sigma}{(1+\sigma)}\right) = 0$$

so

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{f_nN + \frac{dk}{d\tau_n}\tau_k f_k + \frac{dN}{d\tau_n}\tau_n f_n\right\}$$

plugging in our results for $\frac{dN}{d\tau_n}$ and $\frac{dk}{d\tau_n}$, this becomes

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{f_nN + \frac{-k}{(1-\tau_n)}\left(\frac{\sigma}{1+\sigma}\right)\tau_k f_k + \frac{-N}{(1-\tau_n)}\frac{\sigma}{(1+\sigma)}\tau_n f_n\right\}$$

or simply

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = \left\{f_nN - \frac{\sigma}{(1-\tau_n)(1+\sigma)}\left(\tau_k k f_k + \tau_n N f_n\right)\right\}$$

Pull out f_nN

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_nN\left\{1 - \frac{\sigma}{(1-\tau_n)(1+\sigma)}\left(\tau_k\frac{kf_k}{f_nN} + \tau_n\right)\right\}$$

using our expressions for α and $(1-\alpha)$, this becomes

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_nN\left\{1 - \frac{\sigma}{(1-\tau_n)(1-\alpha)(1+\sigma)}\left(\alpha\tau_k + (1-\alpha)\tau_n\right)\right\}$$

or simply

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_nN\left\{1 - \frac{\alpha\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_n)}\frac{\sigma}{1+\sigma}\right\}$$

5.1 Gross and Net capital share and el. of sub.

Now, here

$$\alpha = \frac{kf_k}{f}$$

while we define the gross capital share ψ:

$$\psi = \frac{k\left(f_k + \delta\right)}{f + \delta k}$$

So,

$$\begin{aligned}
\frac{\psi}{\alpha} &= \frac{f}{f+\delta k}\frac{f_k+\delta}{f_k} \\
\frac{\psi}{\alpha} &= \left(\frac{1}{1+\frac{\delta k}{f}}\right)\left(1+\frac{\delta}{f_k}\right)
\end{aligned}$$

Now,

$$\begin{aligned}
f_k &= \frac{\rho+\gamma g}{\left(1-\tau_k\right)} \\
f_k + \delta &= \frac{\rho+\gamma g+\delta\left(1-\tau_k\right)}{\left(1-\tau_k\right)} \\
\frac{\delta k}{f} &= \frac{\delta\alpha}{f_k} = \frac{\delta\alpha\left(1-\tau_k\right)}{\rho+\gamma g}
\end{aligned}$$

So,

$$\begin{aligned}
\psi &= \left(\frac{1}{1+\frac{\delta\alpha(1-\tau_k)}{\rho+\gamma g}}\right)\left(1+\frac{\delta\left(1-\tau_k\right)}{\rho+\gamma g}\right)\alpha \\
\psi &= \left(\frac{\rho+\gamma g+\delta\left(1-\tau_k\right)}{\rho+\gamma g+\alpha\delta\left(1-\tau_k\right)}\right)\alpha
\end{aligned}$$

$$\begin{aligned}
\psi\left(\rho+\gamma g+\alpha\delta\left(1-\tau_k\right)\right) &= \left(\rho+\gamma g+\delta\left(1-\tau_k\right)\right)\alpha \\
\psi\left(\rho+\gamma g\right) &= \left(\rho+\gamma g+\delta\left(1-\tau_k\right)-\delta\psi\left(1-\tau_k\right)\right)\alpha \\
\alpha &= \frac{\psi\left(\rho+\gamma g\right)}{\left(\rho+\gamma g\right)+\delta\left(1-\tau_k\right)-\delta\psi\left(1-\tau_k\right)} \\
\alpha &= \frac{\psi\left(\rho+\gamma g\right)}{\left(\rho+\gamma g\right)+\delta\left(1-\tau_k\right)\left(1-\psi\right)}
\end{aligned}$$

Say $\delta = .03$

$$\alpha = \frac{\psi\left(0.07\right)}{\left(0.07\right)+0.03\left(0.75\right)\left(1-\psi\right)}$$

So if $\psi = 0.33$, $\alpha = 0.27$

Also,

$$\xi = \frac{f_n f_k}{f f_{kn}}$$

while we define the gross elasticity ε

$$\varepsilon = \frac{f_n \left(f_k + \delta \right)}{\left(f + \delta k \right) f_{kn}}$$

So,

$$\frac{\xi}{\varepsilon} = \frac{f + \delta k}{f} \frac{f_k}{f_k + \delta}$$

$$\begin{aligned} f_k &= \frac{\rho + \gamma g}{(1 - \tau_k)} \\ f_k + \delta &= \frac{\rho + \gamma g + \delta (1 - \tau_k)}{(1 - \tau_k)} \\ \frac{\delta k}{f} &= \frac{\delta \alpha}{f_k} = \frac{\delta \alpha (1 - \tau_k)}{\rho + \gamma g} \end{aligned}$$

so

$$\begin{aligned} \frac{\xi}{\varepsilon} &= \frac{\rho + \gamma g + \alpha \delta (1 - \tau_k)}{\rho + \gamma g} \frac{\rho + \gamma g}{\rho + \gamma g + \delta (1 - \tau_k)} \\ \xi &= \left(\frac{\rho + \gamma g + \alpha \delta (1 - \tau_k)}{\rho + \gamma g + \delta (1 - \tau_k)} \right) \varepsilon \end{aligned}$$

Say $\delta = .03$. If $\varepsilon = 1$, then $\alpha = 0.2745$ and we can calculate that $\xi = 0.82$.

$$\xi = \left(\frac{0.07 + (0.2745)(0.03)(0.75)}{0.07 + (0.03)(0.75)} \right) \varepsilon$$

If $f = k^{\psi} n^{1-\psi} - \delta k$, then

$$\xi = \frac{f_n f_k}{f f_{kn}}$$

$$\begin{aligned} \xi &= \frac{(1 - \psi) k^{\psi} n^{-\psi} \left(\psi k^{\psi - 1} n^{1-\psi} - \delta \right)}{\left(k^{\psi} n^{1-\psi} - \delta k \right) \psi (1 - \psi) k^{\psi - 1} n^{-\psi}} \\ \xi &= \frac{\psi k^{\psi - 1} n^{1-\psi} - \delta}{\psi k^{\psi - 1} n^{1-\psi} - \psi \delta} \\ \xi &= \frac{\psi \frac{k}{n}^{\psi - 1} - \delta}{\psi \frac{k}{n}^{\psi - 1} - \psi \delta} \end{aligned}$$

5.2 Showing that capital tax cut has bigger feedback

Take the two results from above,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} \\ -\frac{\alpha\tau_k+(1-\alpha)\tau_n}{(\rho+\gamma g)-\alpha g(1-\tau_k)} \\ \cdot\frac{(\rho+\gamma g)(1-\xi)+(1-\tau_k)(\xi-\alpha)g}{(1-\alpha)(1-\tau_k)}\frac{\sigma}{1+\sigma} \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_n N \left\{ 1 - \frac{\alpha\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_n)}\frac{\sigma}{1+\sigma} \right\}$$

Now, let both tax rates equal τ, and obtain

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{(\alpha+\xi-1)\tau+(1-\alpha)\tau}{(1-\alpha)(1-\tau)} \\ -\frac{\alpha\tau+(1-\alpha)\tau}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{(\rho+\gamma g)(1-\xi)+(1-\tau)(\xi-\alpha)g}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_n N \left\{ 1 - \frac{\alpha\tau + (1-\alpha)\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \right\}$$

Simplifying,

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{\xi\tau}{(1-\alpha)(1-\tau)} \\ -\frac{\tau}{(1-\alpha)(1-\tau)}\frac{(\rho+\gamma g)(1-\xi)+(1-\tau)(\xi-\alpha)g}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{\sigma}{1+\sigma} \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_n N \left\{ 1 - \frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \right\}$$

or

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{\xi\tau}{(1-\alpha)(1-\tau)} \\ -\frac{\tau}{(1-\alpha)(1-\tau)}\frac{(\rho+\gamma g)-\alpha g(1-\tau)+(1-\tau)\xi g-\xi(\rho+\gamma g)}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{\sigma}{1+\sigma} \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_n N \left\{ 1 - \frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \right\}$$

or

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{\xi\tau}{(1-\alpha)(1-\tau)} \\ -\frac{\tau}{(1-\alpha)(1-\tau)}\left(1+\xi\frac{(1-\tau)g-(\rho+\gamma g)}{(\rho+\gamma g)-\alpha g(1-\tau)}\right)\frac{\sigma}{1+\sigma} \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_n N \left\{ 1 - \frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \right\}$$

or, rearranging

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = f_k k \left\{ \begin{array}{c} 1 - \frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \\ -\xi\frac{\tau}{(1-\alpha)(1-\tau)}\left(1-\frac{(\rho+\gamma g)-g(1-\tau)}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{\sigma}{1+\sigma}\right) \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_n}\right|_{dynamic} = f_n N \left\{ 1 - \frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \right\}$$

so the claim is that

$$\begin{aligned}
\frac{\left.\frac{dR}{d\tau_k}\right|_{dynamic}}{f_k k} &< \frac{\left.\frac{dR}{d\tau_n}\right|_{dynamic}}{f_n N} \\
\begin{array}{r} 1-\frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma} \\ -\xi\frac{\tau}{(1-\alpha)(1-\tau)}\left(1-\frac{(\rho+\gamma g)-g(1-\tau)}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{\sigma}{1+\sigma}\right) \end{array} &< 1-\frac{\tau}{(1-\alpha)(1-\tau)}\frac{\sigma}{1+\sigma}
\end{aligned}$$

which holds if:

$$-\xi\frac{\tau}{(1-\alpha)(1-\tau)}\left(1-\frac{(\rho+\gamma g)-g(1-\tau)}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{\sigma}{1+\sigma}\right)<0$$

or

$$1-\frac{(\rho+\gamma g)-g(1-\tau)}{(\rho+\gamma g)-\alpha g(1-\tau)}\frac{\sigma}{1+\sigma}>0$$

or

$$1-\frac{(1-\tau)}{(1-\tau)}\frac{\frac{(\rho+\gamma g)}{(1-\tau)}-g}{\frac{(\rho+\gamma g)}{(1-\tau)}-\alpha g}\frac{\sigma}{1+\sigma}>0$$

or

$$1-\frac{\frac{(\rho+\gamma g)}{(1-\tau)}-g}{\frac{(\rho+\gamma g)}{(1-\tau)}-\alpha g}\frac{\sigma}{1+\sigma}>0$$

now

$$r=f_k=\frac{\rho+\gamma g}{1-\tau_k}$$

so this condition is that

$$1-\frac{r-g}{r-\alpha g}\frac{\sigma}{1+\sigma}>0$$

which holds whenever

$$r>g$$

or, in words, whenever the interest rate (the net marginal product of capital) is greater than the growth rate in the economy. This always holds by the transversality condition.

5.3 General production for Results 5-6

A more general form of production function is (and implies):

$$\begin{aligned}
f(k) &= \phi(k) \\
r &= f_k \\
w &= f-kf_k \\
\frac{w}{r} &= \frac{f}{f_k}-k\text{, or}\quad k=\frac{f}{f_k}-\frac{w}{r}
\end{aligned}$$

Then, taking the derivative of both sides with respect to the ratio $\frac{w}{r}$,

$$\begin{aligned}
\frac{dk}{d\frac{w}{r}} &= \frac{(f_k f_k - f f_{kk})\frac{dk}{d\frac{w}{r}}}{f_k f_k} - 1 \\
\frac{dk}{d\frac{w}{r}}\left(1 - \frac{(f_k f_k - f f_{kk})}{f_k f_k}\right) &= -1 \\
\frac{dk}{d\frac{w}{r}} &= -\frac{f_k f_k}{f f_{kk}}
\end{aligned}$$

With constant RTS, we know that:

$$\alpha = \frac{k f_k}{f} \tag{A11}$$

$$(1-\alpha) = f - \frac{k f_k}{f} \tag{4}$$

Then, the elasticity of substitution between k and inelastic $n = 1$ is:

$$\begin{aligned}
\xi &= \frac{dk}{d\frac{w}{r}}\frac{\frac{w}{r}}{k} \\
\xi &= -\frac{f_k f_k}{f f_{kk}}\left(\frac{f}{k f_k} - 1\right)
\end{aligned}$$

$$\begin{aligned}
\xi &= -\frac{f_k f_k}{f f_{kk}}\frac{f - k f_k}{k f_k} \\
\xi &= \frac{k f_k f_k - f f_k}{k f f_{kk}} = \frac{f_k f_k}{f f_{kk}} - \frac{f_k}{k f_{kk}}
\end{aligned}$$

Rearranging these, we can also show that:

$$\xi = \frac{f_k f_k}{f f_{kk}} - \frac{f_k}{k f_{kk}} \tag{5}$$

$$\xi = \frac{f_k f_k}{f f_{kk}}\left(1 - \frac{f}{k f_k}\right) \quad \text{(Greg's BPEA result)} \tag{6}$$

$$\xi = \frac{f_k f_k}{f f_{kk}}\left(1 - \frac{1}{\alpha}\right) = \frac{f_k f_k}{f f_{kk}}\left(\frac{\alpha - 1}{\alpha}\right) = \frac{f_k f_k}{f f_{kk}}\left(\frac{\alpha - 1}{\frac{k f_k}{f}}\right) \tag{7}$$

$$\xi = \frac{f_k f_k f}{k f_k f f_{kk}}(\alpha - 1) = \frac{f_k}{k f_{kk}}(\alpha - 1) \tag{A13}$$

so,

$$\begin{aligned}
\xi &= \frac{f_k}{k f_{kk}}(\alpha - 1) \\
\frac{f(k)}{f_k}\frac{f_{kk}}{f_k} &= \frac{1}{\xi}\frac{(\alpha - 1)}{\alpha} \\
\frac{(\alpha - 1)}{\alpha}\frac{1}{\xi} &= \frac{f f_{kk}}{f_k f_k}
\end{aligned}$$

Substituting these into the general form of result (5):

$$\frac{dR}{d\tau_k} = kf_k \frac{(1-\tau_n)}{(1-\tau_k)} + \frac{\tau_k}{(1-\tau_k)} \frac{(f_k)^2}{f_{kk}}$$

We can simplify this result to obtain:

$$\begin{aligned}
\frac{dR}{d\tau_k} &= \left[\frac{(1-\tau_n)}{(1-\tau_k)} + \frac{\tau_k \xi}{(1-\tau_k)(\alpha-1)}\right] \alpha f(k) \\
\frac{dR}{d\tau_k} &= \left[\frac{(1-\tau_n)(\alpha-1)+\tau_k \xi}{(1-\tau_k)(\alpha-1)}\right] \alpha f(k) \qquad (8) \\
\frac{dR}{d\tau_k} &= \left[\frac{\alpha-1-\tau_n\alpha+\tau_n+\tau_k\xi}{(1-\tau_k)(\alpha-1)}\right] \alpha f(k)
\end{aligned}$$

$$\begin{aligned}
\frac{dR}{d\tau_k} &= \left[\frac{(1-\tau_k)(\alpha-1)-\tau_n\alpha+\tau_n+\tau_k\xi+\tau_k(\alpha-1)}{(1-\tau_k)(\alpha-1)}\right] \alpha f(k) \\
\frac{dR}{d\tau_k} &= \left[1-\frac{\tau_n\alpha-\tau_n-\tau_k\xi-\tau_k(\alpha-1)}{(1-\tau_k)(\alpha-1)}\right] \alpha f(k) \qquad (9) \\
\frac{dR}{d\tau_k} &= \left[1-\frac{\alpha\tau_k+(1-\alpha)\tau_n+\tau_k(\xi-1)}{(1-\tau_k)(1-\alpha)}\right] \alpha f(k) \qquad (5) \\
\frac{dR}{d\tau_k} &= \left[1-\frac{18}{24}\right] \alpha f(k) \qquad (10)
\end{aligned}$$

not the result from the NBER working paper draft, though the numerical estimate is the same, since the starting tax rates were assumed to be the same.

6 Constant-consumption labor supply elasticity

Here, we derive σ, the constant-consumption elasticity of labor supply with respect to the real wage. First, we apply the implicit function theorem to (15)

$$\begin{aligned}
v'(N) &= \frac{-(1-\tau_n)w}{c} \\
cv'(N)+(1-\tau_n)w &= 0 \\
\frac{dN}{dw} &= \frac{(1-\tau_n)}{cv''(N)}
\end{aligned}$$

Then we derive the constant consumption labor elasticity expression from the main text, using result (15) again:

$$\begin{aligned}
\sigma &= \frac{dN}{dw}\frac{w}{N} \\
\sigma &= \frac{(1-\tau_n)}{cv''(N)}\frac{\frac{v'c}{(1-\tau_n)}}{N} \\
\sigma &= \frac{v'(N)}{v''(N)\cdot N}
\end{aligned}$$

This is the result from the main text. Also note that:

$$\begin{aligned}
Nv''(N) + v'(N) &= \frac{v'(N)}{\sigma} + v'(N) \\
Nv''(N) + v'(N) &= v'(N)N\left(\frac{1+\sigma}{N\sigma}\right)
\end{aligned}$$

6.1 Feldstein Effect

Feldstein has suggested that if labor income is divided into wage and benefits, the non-taxed nature of the second will play a role in the response to a tax change. That is, higher labor income taxes will cause a shift away from wage income and toward benefit income. This would increase the power of labor tax cuts to pay for themselves, as a lower labor income tax rate would encourage workers to move toward wage income, which is taxed. Below, we show that the model from Section 1, with inelastic labor supply, can be modified to divide labor income into taxed wage income and nontaxed benefit income. The technique follows that of the model of Section 2.

The steady state conditions will be

$$y = k^{\alpha}$$

$$r = \alpha k^{\alpha}$$

$$w = (1-\alpha)k^{\alpha} - b$$

Note that this wage equation is different, in that the total labor income per unit of labor is still the MPL, but now benefits are subtracted from the MPL to give wage income.

To find the equation corresponding to the labor-supply condition (10) in Section 2, we need to set up the household's maximization in our new setting. We will assume that the household's felicity function takes a CES form with an elasticity of substitution between consumption and benefits of $-\varepsilon$. That is,

$$u = \left(\beta c^{\frac{1+\varepsilon}{\varepsilon}} + (1-\beta)b^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}}$$

The household's dynamic budget constraint also reflects the availability of benefits as consumption:

$$\dot{k} = (1-\tau_n)w + (1-\tau_k)rk + b - c - b + T$$

Note that, unlike in the basic model of Section 1 and the elastic labor supply model of Section 2, w is no longer taken as given by the household, since they choose how much of the marginal product of labor to receive in wage income rather than in the form of benefits. The maximization problem is:

$$H = e^{-\rho t}\left(\beta c^{\frac{1+\varepsilon}{\varepsilon}} + (1-\beta) b^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} + \varphi((1-\tau_n)w + (1-\tau_k)rk + b - c - b + T)$$

First order conditions:

$$\begin{aligned}
FOC_k &: \quad -\dot{\varphi} = (1-\tau_k)\varphi \\
FOC_c &: \quad e^{-\rho t}\beta c^{\frac{1}{\varepsilon}}\left(\beta c^{\frac{1+\varepsilon}{\varepsilon}} + (1-\beta) b^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}-1} = \varphi \\
FOC_b &: \quad e^{-\rho t}(1-\beta) b^{\frac{1}{\varepsilon}}\left(\beta c^{\frac{1+\varepsilon}{\varepsilon}} + (1-\beta) b^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}-1} = (1-\tau_n)\varphi
\end{aligned}$$

The final FOC reflects that the derivative of w with respect to benefits is -1, since for each dollar of benefts that the worker receives, the employer reduces his wage receipts by one dollar.

Combining the last two FOCs yields:

$$b = \left(\frac{\beta(1-\tau_n)}{1-\beta}\right)^{\varepsilon} c$$

Taking the derivative of FOC_c allows us to derive the steady state condition that parallels equation (11):

$$\begin{aligned}
\frac{dFOC_c}{dt} &: \quad -\rho\varphi + \frac{1}{\varepsilon}\frac{\dot{c}}{c}\varphi + \frac{\frac{-1}{1+\varepsilon}\left(\beta\frac{1+\varepsilon}{\varepsilon}\frac{\dot{c}}{c} + (1-\beta)\frac{1+\varepsilon}{\varepsilon}\frac{\dot{b}}{b}\right)\varphi}{\beta c^{\frac{1+\varepsilon}{\varepsilon}} + (1-\beta) b^{\frac{1+\varepsilon}{\varepsilon}}} = -(1-\tau_k)r\varphi \\
& \quad \text{given that } c \text{ and } b \text{ are constant in the steady state,} \\
-\rho &= -(1-\tau_k)r \\
r &= \frac{\rho}{1-\tau_k}
\end{aligned}$$

Finally, in the steady state total income is used either for consumption or "consumption" of benefits, so

$$c = k^{\alpha} - b$$

Tax revenue, as before is

$$R = \tau_k r k + \tau_n w$$

Combining these steady state conditions, we can solve for the steady state levels of k and b. Specifically,

$$\begin{aligned} k^* &= \left(\frac{\rho}{\alpha(1-\tau_k)}\right)^{\frac{1}{\alpha-1}} \\ b^* &= \frac{\tilde{\beta}}{1+\tilde{\beta}} {k^*}^{\alpha} \end{aligned}$$

where

$$\tilde{\beta} = \left(\frac{\beta(1-\tau_n)}{1-\beta}\right)^{\varepsilon}$$

Then, we can solve for the dynamic effect of a labor tax change on revenue:

$$\frac{dR}{d\tau_n} = (1-\alpha)k^{\alpha} - b + \frac{dk^*}{d\tau_n}(\cdot) + \frac{db^*}{d\tau_n}(-\tau_n)$$

Now, $\frac{dk}{d\tau_n} = 0$ in the steady state, as is clear by the expression for k^* above, which excludes τ_n. But, from the expression for b in the steady state,

$$\frac{db^*}{d\tau_n} = \frac{(1+\tilde{\beta})\left(\frac{-\varepsilon\beta}{1-\beta}\right)\left(\frac{\beta(1-\tau_n)}{1-\beta}\right)^{\varepsilon-1} - \tilde{\beta}\left(\frac{-\varepsilon\beta}{1-\beta}\right)\left(\frac{\beta(1-\tau_n)}{1-\beta}\right)^{\varepsilon-1}}{(1+\tilde{\beta})^2}$$

$$\frac{db^*}{d\tau_n} = \frac{-\varepsilon\left(\frac{\beta}{1-\beta}\right)^{\varepsilon}(1-\tau_n)^{\varepsilon-1}}{(1+\tilde{\beta})^2}$$

Thus, we can write the change in tax revenue as

$$\frac{dR}{d\tau_n} = (1-\alpha)k^{\alpha} - b + \frac{\varepsilon\left(\frac{\beta}{1-\beta}\right)^{\varepsilon}\tau_n(1-\tau_n)^{\varepsilon-1}}{(1+\tilde{\beta})^2}$$

Recall that the elasticity of substitution between b and c is $-\varepsilon$. We want to derive the elasticity of taxable income with respect to $(1-\tau_n)$, the elasticity Feldstein highlights.

Taxable income is $k^{\alpha} - b$

$$\begin{aligned} \frac{d(k^{\alpha}-b)}{d(1-\tau_n)}\frac{(1-\tau_n)}{(k^{\alpha}-b)} &= \frac{d}{d(1-\tau_n)}\left\{k^{\alpha}\left[1 - -\frac{\tilde{\beta}}{1+\tilde{\beta}}\right]\right\}\frac{(1-\tau_n)}{k^{\alpha}\left[1-\frac{\tilde{\beta}}{1+\tilde{\beta}}\right]} \\ \text{this works out to } \Delta &= \frac{-\varepsilon\left(\frac{\beta}{1-\beta}\right)^{\varepsilon}(1-\tau_n)^{\varepsilon}}{(1+\tilde{\beta})} = \frac{-\tilde{\beta}}{1+\tilde{\beta}}\varepsilon \end{aligned}$$

Thus, we can write,

$$\begin{aligned}
\frac{dR}{d\tau_n} &= k^{\alpha}\left((1-\alpha)-\frac{\tilde{\beta}}{1+\tilde{\beta}}\right)\left\{1+\frac{\tau_n\tilde{\beta}\varepsilon}{(1+\tilde{\beta})^2}\frac{1}{(1-\alpha)-\frac{\tilde{\beta}}{1+\tilde{\beta}}}\right\} \\
\frac{dR}{d\tau_n} &= k^{\alpha}\left((1-\alpha)-\frac{\tilde{\beta}}{1+\tilde{\beta}}\right)\left\{1+\frac{\tau_n\tilde{\beta}\varepsilon}{(1+\tilde{\beta})^2}\frac{1+\tilde{\beta}}{\left[(1+\tilde{\beta})(1-\alpha)-\tilde{\beta}\right]}\right\} \\
\frac{dR}{d\tau_n} &= \left.\frac{dR}{d\tau_n}\right|_{static}\left[1+\frac{\tau_n}{(1-\tau_n)}\frac{1}{\left[(1+\tilde{\beta})(1-\alpha)-\tilde{\beta}\right]}\Delta\right]
\end{aligned}$$

7 Finite Horizons: Rule of Thumb Generalization

Consider the two groups of households. For the portion $(1-\lambda)$ that maximize as infinite-horizon households, result (3) from the main text will apply:

$$r=\gamma\frac{\dot{c}}{c}+\rho+\gamma g$$

Now let $\mathbf{C}$ represent aggregate consumption per efficiency unit ("aggregate" in the sense that it includes both infinite-horizon and rule-of-thumb households"). Then, $\frac{\dot{\mathbf{C}}}{\mathbf{C}}$ will be the weighted sum of the growth rates of consumption for the two groups of households. We have $(1-\lambda)$ households increasing consumption as above, and λ households increasing consumption one-for-one with wage income.

$$\begin{aligned}
\frac{\dot{\mathbf{C}}}{\mathbf{C}} &= \lambda\frac{\dot{w}}{w}+(1-\lambda)\left(\frac{r-\rho-\gamma g}{\gamma}\right) \\
r &= \frac{\gamma}{1-\lambda}\left(\frac{\dot{c}}{c}-\lambda\frac{\dot{w}}{w}\right)+\rho+\gamma g
\end{aligned}$$

In the steady state, $\frac{\dot{c}}{c}$ will be constant. Additionally, using our equation (2) from the model of Section I, we can show

$$
\begin{aligned}
w &= f(k) - kf'(k) \\
\frac{\dot{w}}{w} &= \frac{\dot{k}f'(k) - \dot{k}f'(k) - \dot{k}kf''(k)}{f(k) - kf'(k)} \\
\frac{\dot{w}}{w} &= \frac{-\dot{k}kf''(k)}{f(k) - kf'(k)} \\
\frac{\dot{w}}{w} &= \frac{\frac{-\dot{k}kf''(k)}{f(k)}}{1-\alpha} \\
\frac{\dot{w}}{w} &= \frac{-vkf''(k)}{f(k)(1-\alpha)} = \frac{-\dot{k}kf'f''}{ff'(1-\alpha)} \\
\frac{\dot{w}}{w} &= \frac{-\alpha\dot{k}f''}{f'(1-\alpha)} = \frac{-\alpha\dot{k}(\alpha-1)f'}{(1-\alpha)f\alpha\sigma} \\
\frac{\dot{w}}{w} &= \frac{\dot{k}f'}{f\sigma} = \frac{\alpha}{\sigma}\frac{\dot{k}}{k} \\
\frac{\dot{w}}{w} &= \frac{1}{\sigma}\frac{\dot{y}}{y} = \frac{1}{\sigma}\frac{\dot{c}}{c}
\end{aligned}
$$

Assuming Cobb-Douglas production, $\sigma = 1$.

$$
\frac{\dot{w}}{w} = \frac{\dot{\hat{y}}_t}{y_t} = \frac{\dot{c}}{c}
$$

Thus, we can derive:

$$
\begin{aligned}
r &= \frac{\gamma}{1-\lambda}\left(\frac{\dot{c}}{c}(1-\lambda)\right) + \rho + \gamma g \\
r &= \gamma\left(\frac{\dot{c}}{c}\right) + \rho + \gamma g
\end{aligned}
$$

which is identical to the Euler equation without rule of thumb consumers (result (3)).

In the non-Cobb-Douglas case, this result is

$$
r = \gamma\left(\frac{\dot{c}}{c}\right)\frac{1-\frac{\lambda}{\sigma}}{1-\lambda} + \rho + \gamma g
$$

8 Finite Horizons: Derivation of results (16)-(17)

We provide a brief review of the Blanchard model here, as well as deriving the key results of the main text.

Production in the Blanchard model is identical to the Ramsey model, so we simply restate the key relationships from Section 1 for reference.

$$y = f(k)$$
$$\tilde{r} = (1-\tau)f'(k)$$
$$w = f(k) - kf'(k)$$

The derivation of the Blanchard model's Euler equation follows Ramsey except for his infinite-horizon assumption. Let the probability that a household ends be p per period. Therefore, following the procedure of Section 1, we can write the present value of the household i's utility as:

$$\begin{aligned} Utility &= \int_0^\infty e^{-(\rho+p)t} u(c_i(t))dt \\ \text{assuming CRRA,} &= \int_0^\infty e^{-(\rho+p)t} \frac{c_i(t)^{1-\gamma}}{1-\gamma} dt \end{aligned}$$

The after-tax dynamic budget constraint is now:

$$\dot{k}_i = (1-\tau_n)w + ((1-\tau_k)r + p)\, k_i - c_i$$

Note the extra rate of return that the household obtains on k_i. This is due to Blanchard's assumption that assets are annuitized in the economy to prevent accidental bequests. We will discuss this assumption and the impact of relaxing it later in the paper and in this Appendix. To allow temporary indebtedness, the credit markets will require that the present value of household assets, using the augmented discount rate, must be non-negative. That is:

$$\lim_{t\to\infty} k_i(t) e^{-(r+p)t} \geq 0$$

Proceeding as in the Ramsey model, we can set up a Hamiltonian function and find the expression for the household's optimal growth rate of consumption.

$$\begin{aligned} H &= e^{-(\rho+p)t} \frac{c_i(t)^{1-\gamma}}{1-\gamma} + \varphi(t)\left[(1-\tau_n)w + ((1-\tau_k)r + p)\, k_i - c_i + T\right] \\ FOC_k &: ((1-\tau_k)r + p)\, \varphi(t) = -\dot{\varphi}(t) \\ FOC_{c_i} &: e^{-(\rho+p)t} c_i(t)^{-\gamma} = \varphi(t) \\ \frac{dFOC_{c_i}}{dt} &: \dot{\varphi}(t) = -(\rho+p)e^{-(\rho+p)t}c_i(t)^{-\gamma} - \gamma e^{-(\rho+p)t} c_i(t)^{-\gamma-1}\dot{c}(t) \\ -((1-\tau_k)r + p)\, e^{-(\rho+p)t} c_i(t)^{-\gamma} &= -(\rho+p)e^{-(\rho+p)t}c_i(t)^{-\gamma} - \gamma e^{-(\rho+p)t} c_i(t)^{-\gamma-1}\dot{c}(t) \\ -((1-\tau_k)r + p) &= -(\rho+p) - \gamma \frac{\dot{c}_i(t)}{c_i(t)} \\ ((1-\tau_k)r + p) &= \gamma \frac{\dot{c}_i(t)}{c_i(t)} + (\rho+p) \\ \tilde{r} &= \gamma \frac{\dot{c}_i}{c_i} + \rho. \end{aligned}$$

This gives the individual household's change over time in consumption. In the Ramsey model, where all households were infinitely-lived and identical to the aggregate, this would also have described the evolution of aggregate consumption.

However, to derive aggregate consumption dynamics for the Blanchard model, it is not sufficient to use an individual household's plan, since households terminate while society does not. The aggregate consumption dynamics are derived in Barro & Sala-i-Martin (1995, pp 110ff) and result in the following, for a CRRA instantaneous utility function:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma}\left[(\tilde{r}-\rho)-(p+n)\frac{k(t)}{c(t)}\left(\rho+\gamma p-(1-\gamma)\tilde{r}\right)\right]$$

In the steady state, $\frac{\dot{c}_t}{c_t} = g$. Denoting steady state values with a *, we have

$$\begin{aligned}
(\tilde{r}-\rho) &= (p+n)\frac{k(t)^*}{c(t)^*}\left(\rho+\gamma p-(1-\gamma)\tilde{r}\right)+\gamma g \\
\tilde{r} &= \frac{(p+n)k(t)^*(\rho+\gamma p)+(\rho)c(t)^*+\gamma g c(t)}{(p+n)k(t)^*(1-\gamma)+c(t)^*}
\end{aligned}$$

We also note the expression for the change in the capital stock, k

$$\dot{k} = f(k) - c$$

In the steady state, $\frac{\dot{k}}{k} = g$, so

$$c(t)^* = f(k^*) - gk(t)^*$$

We substitute to derive the Euler steady state condition

$$\begin{aligned}
\tilde{r} &= \frac{(p+n)k(t)^*(\rho+\gamma p)+(\rho)\left(f(k^*)-gk(t)^*\right)+\gamma g(f-gk)}{(p+n)k(t)^*(1-\gamma)+f(k^*)-gk(t)^*} \\
\tilde{r} &= \frac{pk(t)^*(\rho+\gamma p)+(\rho+\gamma g)(f(k^*)-gk)}{pk(t)^*(1-\gamma)+f(k^*)-gk(t)^*} \\
\tilde{r} &= \frac{\rho f(k)+k(t)(p)(\rho+\gamma p)+g(\gamma(f(k)-gk)-\rho k)}{f(k)+k(t)(p)(1-\gamma)-gk}.
\end{aligned}$$

For simplicity, we will assume that $g = 0$.

$$\tilde{r} = \frac{\rho f(k)+k(t)(p)(\rho+\gamma p)}{f(k)+k(t)(p)(1-\gamma)}.$$

Assuming $\gamma = 1$, this simplifies to result (16) from the main text

$$r = \frac{1}{1-\tau_k}\left(\rho+p(\rho+p)\frac{k(t)}{f(k)}\right). \tag{16}$$

Tax revenue can be expressed as before:

$$R = \tau_k k f'(k) + \tau_n (f - k f'(k))$$

The derivative of revenue with respect to a change in the capital tax is thus:

$$\frac{dR}{d\tau_k} = kf' + \left[\tau_k \left(kf'' + f'\right) + \tau_n(-kf'')\right] \frac{dk}{d\tau_k}$$

As with the Ramsey model, we have two conditions on steady state $\tilde{r}$. Setting them equal, continuing to assume $g = 0$ for simplicity, we get

$$(1 - \tau_k) f f' = \rho f + p(\rho + p) k$$

With this result and assuming Cobb-Douglas production, we can solve for $\frac{dk}{d\tau_k}$.,

$$\begin{aligned}
(1 - \tau_k) k^{\alpha} \alpha k^{\alpha - 1} &= \rho k^{\alpha} + p(\rho + p) k \\
\alpha(1 - \tau_k) &= \rho k^{1-\alpha} + p(\rho + p) k^{2-2\alpha} \\
0 &= p(\rho + p) k^{2-2\alpha} + \rho k^{1-\alpha} - \alpha(1 - \tau_k)
\end{aligned}$$

With depreciation,

$$\begin{aligned}
(1 - \tau_k)\left(k^{\alpha} - \delta k\right)\left(\alpha k^{\alpha-1} - \delta\right) &= \rho\left(k^{\alpha} - \delta k\right) + p(\rho + p) k \\
(1 - \tau_k)\left(k^{\alpha} - \delta k\right)\left(\alpha k^{\alpha-1} - \delta\right) &= \rho\left(k^{\alpha} - \delta k\right) + p(\rho + p) k \\
(1 - \tau_k)\left(\alpha k^{2\alpha-1} - \delta k^{\alpha} - \alpha\delta k^{\alpha} + \delta^2 k\right) &= \rho k^{\alpha} - \rho\delta k + p(\rho + p) k \\
\left(p(\rho + p) - \delta\left[\rho + \delta(1 - \tau_k)\right]\right) k^{2-2\alpha} + \left[\rho + \delta\left(1 + \alpha\right)\left(1 - \tau_k\right)\right] k^{1-\alpha} - \alpha(1 - \tau_k) &= 0
\end{aligned}$$

which is a quadratic equation in β where

$$\beta = k^{1-\alpha}$$

Solving the quadratic yields:

$$\beta = \frac{-\rho + \sqrt{\rho^2 + 4p(\rho + p)\alpha(1 - \tau_k)}}{2p(\rho + p)}$$

so that

$$k = \left(\frac{-\rho + \sqrt{\rho^2 + 4p(\rho + p)\alpha(1 - \tau_k)}}{2p(\rho + p)}\right)^{\frac{1}{1-\alpha}}$$

with depreciation,

$$\beta = \frac{-\left[\rho + \delta\left(1 + \alpha\right)\left(1 - \tau_k\right)\right] + \sqrt{\left[\rho + \delta\left(1 + \alpha\right)\left(1 - \tau_k\right)\right]^2 + 4\left(p(\rho + p) - \delta\left[\rho + \delta(1 - \tau_k)\right]\right)\alpha(1 - \tau_k)}}{2\left(p(\rho + p) - \delta\left[\rho + \delta(1 - \tau_k)\right]\right)}$$

so that

$$k = \left(\frac{-\left[\rho + \delta\left(1+\alpha\right)\left(1-\tau_k\right)\right] + \sqrt{\left[\rho + \delta\left(1+\alpha\right)\left(1-\tau_k\right)\right]^2 + 4\left(p(\rho+p) - \delta\left[\rho + \delta(1-\tau_k)\right]\right)\alpha(1-\tau_k)}}{2\left(p(\rho+p) - \delta\left[\rho+\delta(1-\tau_k)\right]\right)} \right)^{\frac{1}{1-\alpha}}$$

and

$$\frac{dk}{d\tau_k} = \frac{1}{1-\alpha} \frac{-2p(\rho+p)\frac{1}{2}4p(\rho+p)\alpha\left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{-1}{2}}}{2p(\rho+p)2p(\rho+p)} \left(\frac{-\rho + \sqrt{\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)}}{2p(\rho+p)} \right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{dk}{d\tau_k} = \frac{-\alpha}{1-\alpha} \frac{1}{\left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}} \left(\frac{-\rho + \left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}}{2p(\rho+p)} \right)^{\frac{\alpha}{1-\alpha}}$$

Substituting this result into $\frac{dR}{d\tau_k}$ yields the result in the text:

$$\frac{dR}{d\tau_k} = kf' + \left[\tau_k\left(kf'' + f'\right) + \tau_n(-kf'')\right] \frac{-\alpha}{1-\alpha} \frac{1}{\left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}} \left(\frac{-\rho + \left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}}{2p(\rho+p)} \right)$$

Simplify with Cobb-Douglas

$$\frac{dR}{d\tau_k} = \left\{ \begin{array}{c} k\alpha k^{\alpha-1} + \left[\begin{array}{c} \tau_k\left(k\alpha\left(\alpha-1\right)k^{\alpha-2} + \alpha k^{\alpha-1}\right) \\ +\tau_n(-k\alpha\left(\alpha-1\right)k^{\alpha-2}) \end{array} \right] \frac{-\alpha}{1-\alpha} \\ \frac{1}{\left[\rho^2+4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}} \left(\frac{-\rho + \left[\rho^2+4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}}{2p(\rho+p)} \right)^{\frac{\alpha}{1-\alpha}} \end{array} \right\}$$

$$\frac{dR}{d\tau_k} = \left\{ \begin{array}{c} k\alpha k^{\alpha-1} - \frac{\alpha}{1-\alpha}\alpha k^{\alpha-1}\left[\alpha\tau_k + \left(1-\alpha\right)\tau_n\right] \\ \frac{1}{\left[\rho^2+4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}} \left(\frac{-\rho + \left[\rho^2+4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}}{2p(\rho+p)} \right)^{\frac{\alpha}{1-\alpha}} \end{array} \right\}$$

Now, note that

$$k^\alpha = \left(\frac{-\rho + \sqrt{\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)}}{2p(\rho+p)} \right)^{\frac{\alpha}{1-\alpha}}$$

so,

$$\frac{dR}{d\tau_k} = \left\{ k\alpha k^{\alpha-1} - \frac{\alpha}{1-\alpha}\alpha k^{\alpha-1}\left[\alpha\tau_k + \left(1-\alpha\right)\tau_n\right] \frac{1}{\left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}} k^\alpha \right\}$$

$$\frac{dR}{d\tau_k} = \alpha k^\alpha \left\{ 1 - \frac{\alpha}{1-\alpha}\left[\alpha\tau_k + \left(1-\alpha\right)\tau_n\right] \frac{1}{\left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)\right]^{\frac{1}{2}}} k^{\alpha-1} \right\}$$

Now, note that

$$k^{\alpha-1} = \left(\frac{2p(\rho+p)}{-\rho + \sqrt{\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)}} \right)$$

so

$$\frac{dR}{d\tau_k} = \alpha k^{\alpha} \left\{ 1 - \frac{\alpha}{1-\alpha} \left[\alpha \tau_k + (1-\alpha)\, \tau_n \right] \frac{1}{\left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k) \right]^{\frac{1}{2}}} \left(\frac{2p(\rho+p)}{-\rho + \sqrt{\rho^2 + 4p(\rho+p)\alpha(1-\tau_k)}} \right) \right\}$$

$$\frac{dR}{d\tau_k} = \alpha k^{\alpha} \left\{ 1 - \frac{\alpha}{1-\alpha} \left[\alpha \tau_k + (1-\alpha)\, \tau_n \right] \frac{2p(\rho+p)}{-\rho \left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k) \right]^{\frac{1}{2}} + \left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k) \right]} \right\}$$

which is the result in the main text.

We call the reader's attention to the fact that, if we allow $p = 0$, as in the Ramsey model, which implies $\tilde{r} = \rho$ in this case, (21) reduces to our result (6) from the Ramsey model. You'd need to apply L'hopital's rule to the expression, though, as follows:

$$\frac{dR}{d\tau_k} = \alpha k^{\alpha} \left\{ 1 - \frac{\alpha}{1-\alpha} \left[\alpha \tau_k + (1-\alpha)\, \tau_n \right] \frac{2(\rho+p) + 2p}{-\frac{1}{2}\rho \left(4p + 4\left(\rho + p \right) \right) \alpha(1-\tau_k) \left[\rho^2 + 4p(\rho+p)\alpha(1-\tau_k) \right]^{-\frac{1}{2}} + (4p + \cdots} \right.$$

$\lim_{p \to 0}$ is

$$\frac{dR}{d\tau_k} = \alpha k^{\alpha} \left\{ 1 - \frac{\alpha}{1-\alpha} \left[\alpha \tau_k + (1-\alpha)\, \tau_n \right] \frac{2\rho}{-\frac{1}{2}\rho 4\rho\alpha(1-\tau_k) \left[\rho^2 \right]^{-\frac{1}{2}} + 4\rho\alpha(1-\tau_k)} \right\}$$

$$\frac{dR}{d\tau_k} = \alpha k^{\alpha} \left\{ 1 - \frac{\alpha}{1-\alpha} \left[\alpha \tau_k + (1-\alpha)\, \tau_n \right] \frac{1}{\alpha(1-\tau_k)} \right\}$$

$$\frac{dR}{d\tau_k} = \alpha k^{\alpha} \left\{ 1 - \frac{\left[\alpha \tau_k + (1-\alpha)\, \tau_n \right]}{(1-\alpha)\,(1-\tau_k)} \right\}$$

the Ramsey result.

8.1 Sidebar on Aggregate Euler

In the original appendix, we used the result in Barro & Sala-i-Martin to shortcut to the aggregate Euler equation. Here, we derive it ourselves. The analysis will work off the inelastic labor supply model of Section 1, and we will assume $g = n = 0$ and $\gamma = 1$ for simplicity.

Therefore, the utility function of the household is

$$U = \int e^{-(\rho+p)(v-t)} \frac{c^{1-\gamma} - 1}{1-\gamma} dv \tag{1}$$

subject to the dynamic budget constraint

$$\dot{k}_i = (1-\tau_n)w + ((1-\tau_k)r + p)k_i - c_i + T_i \tag{2}$$

where T_i is transfers given to household i. Equations (1) and (2) imply the individual Euler equation:

$$\frac{\dot{c}}{c} = (1-\tau_k)r - \rho \tag{3}$$

and the tranversality condition is

$$\lim_{v\to\infty} e^{-((1-\tau_k)r+p)(v-t)} k_{jv} = 0 \tag{4}$$

where j indexes the cohort born in period j.

We can integrate the individual dynamic budget constraint to get the individual's lifetime budget constraint:

$$\int c_{jv} e^{-((1-\tau_k)r+p)(v-t)} dv = k_{jt} + (1-\tau_n)\tilde{w}_t + \tilde{T}_{jt} \tag{5}$$

where a tilde over a variable indicates that it is the present value of the stream of that variable over time, i.e.,

$$\tilde{w}_t = \int_t^\infty w_v e^{-((1-\tau_k)r+p)(v-t)} dv \tag{6}$$

From the Euler equation, note that

$$c_v = c_t e^{-((1-\tau_k)r-\rho)(v-t)} \tag{7}$$

Using this and solving the integral in (5), we obtain

$$c_t = (\rho+p)(k_{jt} + (1-\tau_n)\tilde{w}_t + \tilde{T}_{jt}) \tag{8}$$

which states that an individual's marginal propensity to consume out of the present value of wealth is $(\rho+p)$.

Aggregating across cohorts j,

$$C_t = \int_{-\infty}^t c_{jt}(p+n)e^{nj}e^{-p(t-j)} dj \tag{9}$$

$$K_t = \int_{-\infty}^t k_{jt}(p+n)e^{nj}e^{-p(t-j)} dj \tag{10}$$

$$(1-\tau_n)\tilde{W}_t = (1-\tau_n)\tilde{w}e^{nt} = e^{nt}(1-\tau_n)\int_t^\infty w_v e^{-((1-\tau_k)r+p)(v-t)} dv \tag{11}$$

$$\tilde{T}_t = \tilde{T}_{jt} e^{nt} \tag{12}$$

since transfers and wages are independent of age (since the Blanchard probability doesn't depend on age).

Since $(\rho+p)$ is constant, we can simply write

$$C_t = (\rho+p)(K_{jt} + (1-\tau_n)\tilde{W}_t + \tilde{T}_t) \tag{13}$$

so the time derivative is

$$\dot{C}_t = (\rho + p)(\dot{K}_{jt} + (1-\tau_n)\frac{d\tilde{W}_t}{dt} + \frac{d\tilde{T}_t}{dt}) \tag{14}$$

where

$$\dot{K}_t = \frac{d}{dt}\int_{-\infty}^{t} k_{jt}(p+n)e^{nj}e^{-p(t-j)}dj \tag{15}$$

Applying Leibniz's rule to (15), we obtain

$$\begin{aligned}
\dot{K}_t &= k(t,t) + \int_{-\infty}^{t}[\dot{k}_{jt}(p+n)e^{nj}e^{-p(t-j)} - pk_j(p+n)e^{nj}e^{-p(t-j)}]dj \\
&= 0 + [\dot{k}_{jt}e^{nj}e^{-p(t-j)}]_{\infty}^{t} - [pk_j e^{nj}e^{-p(t-j)}]_{\infty}^{t} \\
&= \dot{k}_{jt}e^{nt} - pk_t e^{nt} \\
&= ((1-\tau_k)r + p)K_t + (1-\tau_n)W_t - C_t + T_t - pK_t \\
\dot{K}_t &= (1-\tau_k)rK_t + (1-\tau_n)W_t - C_t + T_t \qquad (16)
\end{aligned}$$

Also,

$$\begin{aligned}
(1-\tau_n)\frac{d\tilde{W}_t}{dt} &= \frac{d}{dt}e^{nt}\int_{t}^{\infty}(1-\tau_n)w_v e^{-((1-\tau_k)r+p)(v-t)}dv \\
(1-\tau_n)\frac{d\tilde{W}_t}{dt} &= ne^{nt}\int_{t}^{\infty}(1-\tau_n)w_t e^{-((1-\tau_k)r+p)(v-t)}dv \\
&\quad + e^{nt}[(1-\tau_n)w_t e^{-((1-\tau_k)r+p)(\infty)} - (1-\tau_n)w_t e^{-((1-\tau_k)r+p)(0)} \\
&\quad + \int_{t}^{\infty}(1-\tau_n)w_v((1-\tau_k)r+p)e^{-((1-\tau_k)r+p)(v-t)}dv] \\
(1-\tau_n)\frac{d\tilde{W}_t}{dt} &= (1-\tau_n)n\tilde{W}_t - (1-\tau_n)w_t e^{nt} + (1-\tau_n)((1-\tau_k)+p)\tilde{W}_t \\
(1-\tau_n)\frac{d\tilde{W}_t}{dt} &= ((1-\tau_k)r+p+n)(1-\tau_n)\tilde{W}_t - (1-\tau_n)w_t e^{nt} \qquad (17)
\end{aligned}$$

And, finally,

$$\frac{d\tilde{T}_t}{dt} = \frac{d}{dt}e^{nt}\int_{t}^{\infty}T_v e^{((1-\tau_k)r+p)(v-t)}dv$$

where

$$T_v = \tau_k r k_v + \tau_n w$$

In parallel to the derivation of $(1-\tau_n)\frac{d\tilde{W}_t}{dt}$, we immediately obtain

$$\begin{aligned}
\frac{d\tilde{T}_t}{dt} &= ((1-\tau_k)r+p+n)\tilde{T}_t - T_t e^{nt} \\
\frac{d\tilde{T}_t}{dt} &= ((1-\tau_k)r+p+n)\tilde{T}_t - \tau_k r K_t + \tau_n w e^{nt} \qquad (18)
\end{aligned}$$

Thus, we can substitute (16)-(18) into the aggregate Euler equation and obtain:

$$\begin{aligned}
\dot{C}_t &= (\rho+p)\left(\dot{K}_{jt} + (1-\tau_n)\frac{d\tilde{W}_t}{dt} + \frac{d\tilde{T}_t}{dt}\right) & (14)\\
&= (\rho+p)\left(\begin{array}{c} (1-\tau_k)rK_t + (1-\tau_n)W_t - C_t + T_v + \\ ((1-\tau_k)r+p+n)(1-\tau_n)\tilde{W}_t - (1-\tau_n)w_t e^{nt} \\ +((1-\tau_k)r+p+n)\tilde{T}_t - \tau_k r K_t - \tau_n w e^{nt} \end{array}\right) \\
&= (\rho+p)\left(\begin{array}{c} (1-\tau_k)rK_t + (1-\tau_n)W_t - C_t + \tau_k r K_t + \tau_n w e^{nt} + \\ ((1-\tau_k)r+p+n)(1-\tau_n)\tilde{W}_t - (1-\tau_n)w_t e^{nt} \\ +((1-\tau_k)r+p+n)\tilde{T}_t - \tau_k r K_t - \tau_n w e^{nt} \end{array}\right) & (11)\\
&= (\rho+p)\left(\begin{array}{c} (1-\tau_k)rK_t - C_t + \\ ((1-\tau_k)r+p+n)(1-\tau_n)\tilde{W}_t \\ +((1-\tau_k)r+p+n)\tilde{T}_t \end{array}\right)
\end{aligned}$$

Now we must substitute in for C.

$$C_t = (\rho+p)(K_{jt} + (1-\tau_n)\tilde{W}_t + \tilde{T}_t) \qquad (13)$$

so,

$$\begin{aligned}
\dot{C}_t &= (\rho+p)\left(\begin{array}{c} (1-\tau_k)rK_t - (\rho+p)(K_{jt} + (1-\tau_n)\tilde{W}_t + \tilde{T}_t) + \\ ((1-\tau_k)r+p+n)\left[(1-\tau_n)\tilde{W}_t + \tilde{T}_t\right] \end{array}\right) \\
\dot{C}_t &= (\rho+p)\left[((1-\tau_k)r-\rho)\,(K_{jt} + (1-\tau_n)\tilde{W}_t + \tilde{T}_t) - pK + n\left((1-\tau_n)\tilde{W}_t + \tilde{T}_t\right)\right]
\end{aligned}$$

therefore,

$$\begin{aligned}
\frac{\dot{C}_t}{C_t} &= \frac{(\rho+p)\left[((1-\tau_k)r-\rho)\,(K_{jt} + (1-\tau_n)\tilde{W}_t + \tilde{T}_t) - pK + n\left((1-\tau_n)\tilde{W}_t + \tilde{T}_t\right)\right]}{(\rho+p)(K_{jt} + (1-\tau_n)\tilde{W}_t + \tilde{T}_t)} \\
\frac{\dot{C}_t}{C_t} &= ((1-\tau_k)r-\rho) - \frac{(\rho+p)pK}{C_t} + \frac{n\left((1-\tau_n)\tilde{W}_t + \tilde{T}_t\right)}{C_t} \\
\frac{\dot{C}_t}{C_t} &= ((1-\tau_k)r-\rho) - \frac{(\rho+p)\,(p+n)\,K}{C_t} + n & (19)
\end{aligned}$$

as in Barro & Sala-i-Martin.

This analysis was completed by noting that consumption equalled output in the steady-state. That is,

$$C^* = f(K^*) \qquad (20)$$

8.2 Aside on Incomplete Annuitization in Blanchard model

As Blanchard (1985) and Yaari (1965) argued, agents in a finite-horizon world with probabilistic death face an uncertain time of death, and if optimizing, would seek to enter into annuity contracts in which an annuity issuer would pay them a per-period premium in exchange for a claim on their assets at the time of death.[1] If the annuity market is competitive, those premia, when paid to a large population, will equal the expected assets assumed by the insurance companies. If the probability of death is p, each agent receives a premium per period of p for each unit of the consumption good that the annuity issuer will assume upon the agent's death. If the annuity market is not complete, there will be unintended bequests by agents who die "early".

If annuitization is incomplete, there will be unintended bequests. We will assume for simplicity that these are transferred in lump-sum fashion to new entrants to the population. An existing household's flow budget constraint can be written as follows. Note tht the rate of return on capital is now simply r, as the lack of annuity markets means that the household no longer enjoys return $r + \rho$.

$$\dot{k}(t) = rk(t) + w(t) - c(t)$$

The household's utility function is the same as before, though we assume log utility in this section for convenience:

$$U = \int_0^\infty e^{-(\rho+p)t} \ln c(t) dt$$

Setting up the Hamiltonian and maximizing, as above, we obtain:

$$\begin{aligned}
\lim_{t\to\infty} e^{-rt}k(t) &= k(0)e^{-r\cdot 0} + \int_0^\infty e^{-rt}(w(t) - c(t))dt \\
\text{by the transversality condition, } \lim_{t\to\infty} e^{-rt}k(t) &= 0, \text{ so} \\
\int_0^\infty e^{(-r+r-\rho-p)t}c(0) &= k(0) + \tilde{w} \\
\int_0^\infty e^{-(\rho+p)t}c(0) &= k(0) + \tilde{w} \\
c(0) &= (\rho + p)\left[k(0) + \tilde{w}(0)\right]
\end{aligned}$$

where we denote the present value of wages $\tilde{w}$.

We will continue to assume $n = 0$ for simplicity, so new households enter the population at rate p to hold population constant. Our aggregate consumption, capital, wealth, and bequests can be written:

$$\begin{aligned}
\mathbf{C}(t) &= \int_{-\infty}^t c(j,t)pe^{-p(t-j)}dj \\
\mathbf{K}(t) &= \int_{-\infty}^t k(j,t)pe^{-p(t-j)}dj
\end{aligned}$$

[1] This structure can alternatively be thought of as the existence of insurance markets, in which the agents pay insurers a per-period premium in exchange for which the insurers cover the agents' debts at the time of death.

where each $c(j,t)$ is multiplied by p, the size of the cohort, and raised to $e^{-p(t-j)}$ to scale by the number of cohort members alive at time $t \geq j$. The present value of aggregate wages at time t can be written as follows, where the discount rate is r because the rate of return is r, not $r+p$, due to the lack of annuitization.

$$\tilde{W}(t) = \int_t^{\infty} w(v)e^{-rv}dv$$

Solving the aggregate consumption equation by substituting in our equation, simplifying, and applying the transversality condition, we get

$$\begin{aligned}
\mathbf{C}(t) &= \int_{-\infty}^{t} c(j,t)pe^{-p(t-j)}dj \\
\mathbf{C}(t) &= \int_{-\infty}^{t} (\rho+p)\left[k(j)+\tilde{w}(j)\right]pe^{-p(t-j)}dj \\
\mathbf{C}(t) &= (\rho+p)\left[\frac{\mathbf{K}(t)+\tilde{W}(t)}{p}pe^{-p(t-t)} - \frac{\mathbf{K}(t)+\tilde{W}(t)}{p}pe^{-p(t+\infty)}\right] \\
\mathbf{C}(t) &= (\rho+p)\left[\mathbf{K}(t)+\tilde{W}(t)\right]
\end{aligned}$$

To derive the aggregate consumption dynamics, we note that

$$\dot{\mathbf{C}}(t) = (\rho+p)\left[\mathbf{K}(t) + \frac{d\tilde{W}(t)}{dt}\right]$$

We can determine $\dot{\mathbf{K}}$ with the following process, which utilizes Leibniz's rule for the derivative of an integral:

$$\begin{aligned}
\dot{\mathbf{K}}(t) &= \frac{d}{dt}\left[\int_{-\infty}^{t} k(j,t)pe^{-p(t-j)}dj\right] \\
\dot{\mathbf{K}}(t) &= k(t,t)pe^{-p(t-t)} + \int_{-\infty}^{t}\left[\dot{k}(j,t)pe^{-p(t-j)} - pk(j,t)pe^{-p(t-j)}\right]dj
\end{aligned}$$

In traditional Blanchard analysis, $k(t,t)=0$, since new households enter with no assets. In this model, they immediately receive the per capita bequest transfer upon entering, so $k(t,t) = b(t) = \frac{pk(t)}{p} = k(t)$. Thus, using this result and $k(t)$ from before, and calculating the integral above, we get:

$$\begin{aligned}
\dot{\mathbf{K}}(t) &= pk(t) + \frac{p}{p}\left[rk(t)+w(t)-c(t)\right] - pk(t) \\
\dot{\mathbf{K}}(t) &= r\mathbf{K}(t) + W(t) - \mathbf{C}(t) \\
\dot{\mathbf{K}}(t) &= r\mathbf{K}(t) + W(t) - \mathbf{C}(t)
\end{aligned}$$

We can determine $\frac{d\tilde{W}(t)}{dt}$ as follows. Note that $e^{nt} = 1$, since $n = 0$ by assumption:

$$
\begin{aligned}
\frac{d\tilde{W}(t)}{dt} &= \frac{d}{dt}\int_t^\infty w(v)e^{-r(v-t)}dv \\
&= -w(t) + \int rw(v)e^{-r(v-t)}dv \\
&= -w(t) + \int r\tilde{w}(v)dv \\
&= -w(t) + r\tilde{w}(t) \\
&= -W(t) + r\tilde{W}(t) \\
\frac{d\tilde{W}(t)}{dt} &= r\tilde{W}(t) - W(t)
\end{aligned}
$$

Inserting results into our equation for $\dot{C}(t)$, we get:

$$
\begin{aligned}
\dot{\mathbf{C}}(t) &= (\rho + p)\left[\dot{\mathbf{K}}(t) + \frac{d\tilde{W}(t)}{dt}\right] \\
\dot{\mathbf{C}}(t) &= (\rho + p)\left[r\mathbf{K}(t) + W(t) - \mathbf{C}(t) + r\tilde{W}(t) - W(t)\right] \\
\dot{\mathbf{C}}(t) &= (\rho + p)\left[r\mathbf{K}(t) - \mathbf{C}(t) + r\tilde{W}(t)\right]
\end{aligned}
$$

Dividing by $\mathbf{C}(t)$ to obtain the growth rate of per capita aggregate consumption, and simplifying,

$$
\begin{aligned}
\frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)} &= \frac{(\rho + p)\left[r\mathbf{K}(t) - \mathbf{C}(t) + r\tilde{W}(t)\right]}{(\rho + p)\left[\mathbf{K}(t) + \tilde{W}(t)\right]} \\
\frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)} &= \frac{rK(t) - \mathbf{C}(t) + r\tilde{W}(t)}{\mathbf{K}(t) + \tilde{W}(t)} \\
& \quad \text{substituting in } \mathbf{C}(t) \text{ to the numerator} \\
\frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)} &= \frac{r\mathbf{K}(t) - (\rho + p)\left[\mathbf{K}(t) + \tilde{W}(t)\right] + r\tilde{W}(t)}{\mathbf{K}(t) + \tilde{W}(t)} \\
\frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)} &= \frac{(r - \rho - p)\,\mathbf{K}(t) + (r - \rho - p)\,\tilde{W}(t)}{\mathbf{K}(t) + \tilde{W}(t)} \\
\frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)} &= \frac{(r - \rho - p)\left[\mathbf{K}(t) + \tilde{W}(t)\right]}{\mathbf{K}(t) + \tilde{W}(t)} \\
\frac{\dot{\mathbf{C}}(t)}{\mathbf{C}(t)} &= r - \rho - p
\end{aligned}
$$

We point out the similarity between this result and the result for the growth rate of consumption in the Ramsey model, $r - \rho$.

9 Externalities to Capital

The equations are:

$$\kappa = k^{\beta}. \tag{22}$$

$$y = \kappa k^{\alpha} n^{1-\alpha}. \tag{23}$$

$$r = \alpha \kappa k^{\alpha-1} n^{1-\alpha}. \tag{24}$$

$$w = (1-\alpha)\kappa k^{\alpha} n^{-\alpha}. \tag{25}$$

$$v'(n) = \frac{-\left(1-\tau_n\right) w}{c}. \tag{26}$$

$$r = \frac{\rho + \gamma g}{1-\tau_k}. \tag{27}$$

$$c = \kappa k^{\alpha} n^{1-\alpha} - gk. \tag{28}$$

Otherwise, the analysis is exactly as in the main model. To derive these results, we use the system of two equations that simplifies (22-28):In Cobb-Douglas,

$$(1-\tau_k)\alpha k^{\alpha+\beta-1} N^{1-\alpha} - (\gamma g + \rho) = 0.$$

$$v'(N) \cdot \left(k^{\alpha+\beta} N^{1-\alpha} - gk\right) + \left(1-\tau_n\right)(1-\alpha) k^{\alpha+\beta} n^{-\alpha} = 0.$$

9.1 Capital Tax Cut

For $\frac{dk}{d\tau_k}$, take the total derivative of the first of these:

$$d\tau_k \left[-\alpha k^{\alpha+\beta-1} N^{1-\alpha}\right] + d\tau_n [0] + dN \left[(1-\tau_k)\alpha (1-\alpha) k^{\alpha+\beta-1} N^{-\alpha}\right] + dk \left[(1-\tau_k)\alpha (\alpha+\beta-1) k^{\alpha+\beta-2} N^{1-\alpha}\right]$$

$$\frac{dk}{d\tau_k} = \frac{\alpha k^{\alpha+\beta-1} N^{1-\alpha} - \frac{dN}{d\tau_k}\left[(1-\tau_k)\alpha (1-\alpha) k^{\alpha+\beta-1} N^{-\alpha}\right]}{(1-\tau_k)\alpha (\alpha+\beta-1) k^{\alpha+\beta-2} N^{1-\alpha}}$$

For $\frac{dN}{d\tau_k}$, again apply the implicit function theorem, this time to the second of these

$$\begin{aligned} v'(N) \cdot (f(k,N) - gk) + (1-\tau_n) f_n(k,N) &= 0 \\ \text{We switch to Cobb-Douglas, where } f(k,N) &= k^{\alpha+\beta} N^{1-\alpha} \\ v'(N)\left[k^{\alpha+\beta} N^{1-\alpha} - gk\right] + (1-\tau_n)(1-\alpha) k^{\alpha+\beta} N^{-\alpha} &= 0 \end{aligned}$$

From the first, we can write

$$\begin{aligned} \alpha k^{\alpha+\beta-1} N^{1-\alpha} &= \frac{\rho + \gamma g}{1-\tau_k} \\ z &= \frac{k}{n} = \left(\frac{\rho + \gamma g}{\alpha \kappa (1-\tau_k)}\right)^{\frac{1}{\alpha-1}} \end{aligned}$$

We can rewrite the second with z.

$$\begin{aligned}
v'(N) &= \frac{-(1-\tau_n)(1-\alpha)\kappa z^{\alpha}}{\kappa z^{\alpha}N - gzN} \\
v'(N)N &= \frac{-(1-\tau_n)(1-\alpha)\kappa}{\kappa - gz^{1-\alpha}} \\
v'(N)N &= \frac{-(1-\tau_n)(1-\alpha)\kappa}{\kappa - \frac{\alpha\kappa(1-\tau_k)g}{\rho+\gamma g}} \\
v'(N)N &= \frac{-(1-\tau_n)(1-\alpha)}{1 - \frac{\alpha(1-\tau_k)g}{\rho+\gamma g}} \\
v'(N)N &= \frac{-(1-\tau_n)(1-\alpha)\left(\rho+\gamma g\right)}{\rho + g(\gamma - \alpha(1-\tau_k))}
\end{aligned}$$

Rewriting,

$$v'(N)N + \frac{(1-\tau_n)(1-\alpha)\left(\rho+\gamma g\right)}{\rho + g(\gamma - \alpha(1-\tau_k))} = 0$$

We take the total derivative of this expression to find our result.

$$\left\{ \begin{array}{c} dN\left[Nv''(N) + v'(N)\right] + \\ d\tau_n\left[\frac{[\rho+g(\gamma-\alpha(1-\tau_k))](-)[(1-\alpha)(\rho+\gamma g)]}{[\rho+g(\gamma-\alpha(1-\tau_k))]^2}\right] + \\ d\tau_k\left[\frac{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)\alpha g}{[\rho+g(\gamma-\alpha(1-\tau_k))]^2}\right] \end{array} \right\} = 0.$$

Dividing through by $d\tau_k$, we obtain

$$\frac{dN}{d\tau_k}\left[Nv''(N) + v'(N)\right] = \frac{(1-\tau_n)(1-\alpha)(\rho+\gamma g)\alpha g}{\left[\rho + g(\gamma - \alpha(1-\tau_k))\right]^2}.$$

From our results for the elasticity of labor supply, we know that

$$\begin{aligned}
Nv''(N) + v'(N) &= v'(N)\left(\frac{1+\sigma}{\sigma}\right) \\
Nv''(N) + v'(N) &= \frac{-(1-\tau_n)(1-\alpha)\left(\rho+\gamma g\right)}{\rho + g(\gamma - \alpha(1-\tau_k))}\left(\frac{1+\sigma}{N\sigma}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{dN}{d\tau_k} &= \frac{(1-\tau_n)(1-\alpha)(\rho+\gamma g)\alpha g}{\left[\rho + g(\gamma - \alpha(1-\tau_k))\right]^2}\frac{\rho + g(\gamma - \alpha(1-\tau_k))}{-(1-\tau_n)(1-\alpha)\left(\rho+\gamma g\right)}\left(\frac{N\sigma}{1+\sigma}\right) \\
\frac{dN}{d\tau_k} &= \frac{-\alpha g}{\rho + g(\gamma - \alpha(1-\tau_k))}\left(\frac{N\sigma}{1+\sigma}\right) \\
\frac{dN}{d\tau_k} &= \frac{\alpha g N}{\alpha g(1-\tau_k) - (\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right).
\end{aligned}$$

We use this in our expression for $\frac{dk}{d\tau_k}$ to obtain, assuming Cobb-Douglas.

$$\frac{dk}{d\tau_k} = \frac{\alpha k^{\alpha+\beta-1} N^{1-\alpha} - \frac{dN}{d\tau_k}\left[(1-\tau_k)\alpha\left(1-\alpha\right)k^{\alpha+\beta-1}N^{-\alpha}\right]}{(1-\tau_k)\alpha\left(\alpha+\beta-1\right)k^{\alpha+\beta-2}N^{1-\alpha}}$$

$$\frac{dk}{d\tau_k} = \frac{\alpha k^{\alpha+\beta-1} N^{1-\alpha} - \frac{\alpha g N}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)\left[(1-\tau_k)\alpha\left(1-\alpha\right)k^{\alpha+\beta-1}N^{-\alpha}\right]}{(1-\tau_k)\alpha\left(\alpha+\beta-1\right)k^{\alpha+\beta-2}N^{1-\alpha}}$$

Cancelling terms in the numerator and denominator,

$$\frac{dk}{d\tau_k} = \frac{1-(1-\tau_k)\left(1-\alpha\right)\frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)}{(1-\tau_k)\left(\alpha+\beta-1\right)k^{-1}}$$

$$\begin{aligned}
\frac{dk}{d\tau_k} &= \frac{1-(1-\tau_k)\left(1-\alpha\right)\frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)}{(1-\tau_k)\left(\alpha+\beta-1\right)k^{-1}} \\
\frac{dk}{d\tau_k} &= \frac{\left[\alpha g(1-\tau_k)-(\rho+\gamma g)\right](1+\sigma)-(1-\tau_k)(1-\alpha)\alpha g\sigma}{\left[\alpha g(1-\tau_k)-(\rho+\gamma g)\right](1+\sigma)(1-\tau_k)(\alpha+\beta-1)k^{-1}} \\
\frac{dk}{d\tau_k} &= \frac{\left[\alpha g(1-\tau_k)-(\rho+\gamma g)\right](1+\sigma)-(1-\tau_k)(1-\alpha)\alpha g\sigma}{-(1-\tau_k)(1-\alpha-\beta)k^{-1}\left[\alpha g(1-\tau_k)-(\rho+\gamma g)\right](1+\sigma)} \\
\frac{dk}{d\tau_k} &= \frac{k\left\{\alpha(1-\alpha)(1-\tau_k)g\sigma-\left[\alpha g(1-\tau_k)-(\rho+\gamma g)\right](1+\sigma)\right\}}{(1-\alpha-\beta)(1-\tau_k)\left[\alpha g(1-\tau_k)-(\rho+\gamma g)\right](1+\sigma)}
\end{aligned}$$

Though this is not a particularly simple expression, it reduces well in our overall result. That is, insert this result for $\frac{dk}{d\tau_k}$ and our result for $\frac{dN}{d\tau_k}$ into our result for $\frac{dR}{d\tau_k}$ from the main text to obtain, assuming Cobb-Douglas,

$$\begin{aligned}
R &= \tau_k r k + \tau_n w N \\
R &= \left(\alpha\tau_k + (1-\alpha)\tau_n\right)k^{\alpha+\beta}N^{1-\alpha} \\
\left.\frac{dR}{d\tau_k}\right|_{dynamic} &= \left\{\begin{array}{c} \alpha k^{\alpha+\beta}N^{1-\alpha} \\ +\frac{dk}{d\tau_k}(\alpha+\beta)k^{\alpha+\beta-1}N^{1-\alpha}\left(\alpha\tau_k+(1-\alpha)\tau_n\right) \\ +\frac{dN}{d\tau_k}(1-\alpha)k^{\alpha+\beta}N^{-\alpha}\left(\alpha\tau_k+(1-\alpha)\tau_n\right) \end{array}\right\}
\end{aligned}$$

simplifying, and pulling out kf_k, we obtain

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{\begin{array}{c} \alpha k^{\alpha+\beta}N^{1-\alpha}+ \\ \left(\alpha\tau_k+(1-\alpha)\tau_n\right)\left[\begin{array}{c} \frac{dk}{d\tau_k}(\alpha+\beta)k^{\alpha+\beta-1}N^{1-\alpha} \\ +\frac{dN}{d\tau_k}(1-\alpha)k^{\alpha+\beta}N^{-\alpha} \end{array}\right] \end{array}\right\}$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{\begin{array}{c} \alpha k^{\alpha+\beta}N^{1-\alpha} \\ \left(\alpha\tau_k+(1-\alpha)\tau_n\right)\left[\begin{array}{c} \frac{1-(1-\tau_k)(1-\alpha)\frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)}{(1-\tau_k)(\alpha+\beta-1)k^{-1}}(\alpha+\beta)k^{\alpha+\beta-1}N^{1-\alpha} \\ +\frac{\alpha g N}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)(1-\alpha)k^{\alpha+\beta}N^{-\alpha} \end{array}\right] \end{array}\right\}$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \left\{ \begin{array}{c} \alpha k^{\alpha+\beta} N^{1-\alpha} + \\ \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \alpha k^{\alpha+\beta} N^{1-\alpha} \left[\begin{array}{c} \frac{1-(1-\tau_k)(1-\alpha)\frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)}{(1-\tau_k)(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} \\ + \frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \left(\frac{\sigma}{1+\sigma}\right) \frac{(1-\alpha)}{\alpha} \end{array} \right] \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \alpha k^{\alpha+\beta} N^{1-\alpha} \left\{ \begin{array}{c} 1+ \\ \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \left[\begin{array}{c} \frac{1-(1-\tau_k)(1-\alpha)\frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)}{(1-\tau_k)(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} \\ + \frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \left(\frac{\sigma}{1+\sigma}\right) \frac{(1-\alpha)}{\alpha} \end{array} \right] \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \alpha k^{\alpha+\beta} N^{1-\alpha} \left\{ \begin{array}{c} 1+ \\ \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \left[\begin{array}{c} \frac{1}{(1-\tau_k)(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} - \frac{(1-\alpha)\frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)}\left(\frac{\sigma}{1+\sigma}\right)}{(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} \\ + \frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \left(\frac{\sigma}{1+\sigma}\right) \frac{(1-\alpha)}{\alpha} \end{array} \right] \end{array} \right.$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \alpha k^{\alpha+\beta} N^{1-\alpha} \left\{ \begin{array}{c} 1+ \\ \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \left[\begin{array}{c} \frac{1}{(1-\tau_k)(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} + \frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \left(\frac{\sigma}{1+\sigma}\right) \frac{(\alpha+\beta)}{\alpha} \frac{(1-\alpha}{(1-\alpha-} \\ + \frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \left(\frac{\sigma}{1+\sigma}\right) \frac{(1-\alpha)}{\alpha} \end{array} \right. \end{array} \right.$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \alpha k^{\alpha+\beta} N^{1-\alpha} \left\{ \begin{array}{c} 1+ \\ \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \left[\begin{array}{c} \frac{1}{(1-\tau_k)(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} \\ + \frac{\alpha g}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \left(\frac{\sigma}{1+\sigma}\right) \left[\frac{(\alpha+\beta)}{\alpha} \frac{(1-\alpha)}{(1-\alpha-\beta)} + \frac{(1-\alpha)}{\alpha}\right] \end{array} \right] \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \alpha k^{\alpha+\beta} N^{1-\alpha} \left\{ \begin{array}{c} 1+ \\ \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \left[\frac{1}{(1-\tau_k)(\alpha+\beta-1)} \frac{(\alpha+\beta)}{\alpha} + \frac{1}{\alpha g(1-\tau_k)-(\rho+\gamma g)} \frac{\sigma}{1+\sigma} g \left(\frac{(1-\alpha)}{(1-\alpha-\beta)}\right)\right] \end{array} \right\}$$

$$\left.\frac{dR}{d\tau_k}\right|_{dynamic} = \alpha k^{\alpha+\beta} N^{1-\alpha} \left\{ \begin{array}{c} 1- \\ \frac{(\alpha\tau_k+(1-\alpha)\tau_n)}{(1-\tau_k)(1-\alpha-\beta)} \frac{\alpha+\beta}{\alpha} - \frac{(\alpha\tau_k+(1-\alpha)\tau_n)}{(\rho+\gamma g)-\alpha(1-\tau_k)g} \frac{(1-\alpha)}{(1-\alpha-\beta)} \frac{\sigma}{1+\sigma} g \end{array} \right\}$$

9.2 Labor Tax Cut

For $\frac{dN}{d\tau_n}$, we refer to the total derivative of the second. Dividing through by $d\tau_n$, we obtain

$$\left\{ \begin{array}{c} dN\left[Nv''(N) + v'(N)\right] + \\ d\tau_n \left[\frac{[\rho+g(\gamma-\alpha(1-\tau_k))](-)[(1-\alpha)(\rho+\gamma g)]}{[\rho+g(\gamma-\alpha(1-\tau_k))]^2}\right] + \\ d\tau_k \left[\frac{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)\alpha g}{[\rho+g(\gamma-\alpha(1-\tau_k))]^2}\right] \end{array} \right\} = 0.$$

Dividing through by $d\tau_k$, we obtain

$$\frac{dN}{d\tau_n}\left[Nv''(N) + v'(N)\right] = \frac{[(1-\alpha)(\rho+\gamma g)]}{[\rho+g(\gamma-\alpha(1-\tau_k))]}.$$

From our results for the elasticity of labor supply, we know that

$$\begin{aligned} Nv''(N) + v'(N) &= v'(N)\left(\frac{1+\sigma}{\sigma}\right) \\ Nv''(N) + v'(N) &= \frac{-(1-\tau_n)(1-\alpha)\,(\rho+\gamma g)}{\rho + g(\gamma - \alpha(1-\tau_k))}\left(\frac{1+\sigma}{N\sigma}\right). \end{aligned}$$

Thus,

$$\frac{dN}{d\tau_n}\left[Nv''(N) + v'(N)\right] = \frac{\rho + g(\gamma - \alpha(1-\tau_k))}{-(1-\tau_n)(1-\alpha)\,(\rho+\gamma g)} \frac{[(1-\alpha)(\rho+\gamma g)]}{[\rho + g(\gamma - \alpha(1-\tau_k))]} \frac{N\sigma}{(1+\sigma)}.$$

$$\frac{dN}{d\tau_n} = \frac{N}{-(1-\tau_n)} \frac{\sigma}{(1+\sigma)}.$$

Substituting this expression into that for $\frac{dk}{d\tau_n}$ gives

$$d\tau_k\left[-\alpha k^{\alpha+\beta-1}N^{1-\alpha}\right] + d\tau_n\,[0] + dN\left[(1-\tau_k)\alpha\,(1-\alpha)\,k^{\alpha+\beta-1}N^{-\alpha}\right] + dk\left[(1-\tau_k)\alpha\,(\alpha+\beta-1)\,k^{\alpha+\beta-2}N^{1-\alpha}\right]$$

$$\frac{dk}{d\tau_n} = \frac{dN}{d\tau_n}\frac{(1-\alpha)\,k}{(1-\alpha-\beta)N}$$

$$\frac{dk}{d\tau_n} = \frac{-k}{(1-\tau_n)}\frac{\sigma}{(1+\sigma)}\frac{(1-\alpha)}{(1-\alpha-\beta)}$$

Returning to our expression for $\frac{dR}{d\tau_n}$, we obtain

$$\begin{aligned} R &= \tau_k r k + \tau_n w N \\ R &= \left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) k^{\alpha+\beta} N^{1-\alpha} \\ \left.\frac{dR}{d\tau_n}\right|_{dynamic} &= \left\{ \begin{array}{c} (1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha} \\ +\frac{dk}{d\tau_n}\,(\alpha+\beta)\,k^{\alpha+\beta-1}N^{1-\alpha}\left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \\ +\frac{dN}{d\tau_n}\,(1-\alpha)\,k^{\alpha+\beta}N^{-\alpha}\left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \end{array} \right\} \end{aligned}$$

$$\begin{aligned} \left.\frac{dR}{d\tau_n}\right|_{dynamic} &= \left\{ \begin{array}{c} (1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha} \\ +\frac{-k}{(1-\tau_n)}\frac{\sigma}{(1+\sigma)}\frac{(1-\alpha)}{(1-\alpha-\beta)}\,(\alpha+\beta)\,k^{\alpha+\beta-1}N^{1-\alpha}\left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \\ +\frac{N}{-(1-\tau_n)}\frac{\sigma}{(1+\sigma)}\,(1-\alpha)\,k^{\alpha+\beta}N^{-\alpha}\left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \end{array} \right\} \\ \left.\frac{dR}{d\tau_n}\right|_{dynamic} &= \left\{ \begin{array}{c} (1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha} \\ -\left[\frac{1}{(1-\tau_n)}\frac{(1-\alpha)}{(1-\alpha-\beta)}\frac{(\alpha+\beta)}{(1-\alpha)} + \frac{1}{(1-\tau_n)}\right]\frac{\sigma}{(1+\sigma)}\,(1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha}\left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \end{array} \right\} \end{aligned}$$

$$\begin{aligned} \left.\frac{dR}{d\tau_n}\right|_{dynamic} &= \left\{ \begin{array}{c} (1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha} \\ -\left[\frac{1}{(1-\tau_n)}\frac{1}{(1-\alpha-\beta)}\right]\frac{\sigma}{(1+\sigma)}\,(1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha}\left(\alpha\tau_k + (1-\alpha)\,\tau_n\right) \end{array} \right\} \\ \left.\frac{dR}{d\tau_n}\right|_{dynamic} &= (1-\alpha)\,k^{\alpha+\beta}N^{1-\alpha}\left\{1 - \frac{\alpha\tau_k + (1-\alpha)\,\tau_n}{(1-\alpha-\beta)(1-\tau_n)}\frac{\sigma}{(1+\sigma)}\right\} \end{aligned}$$

10 Imperfect Competition–Results (18)-(23)

On the production side, note that

$$f(k,n) = k^{\alpha} n^{1-\alpha} - \delta k. \tag{1}$$

However, due to imperfect competition, factor returns are distorted away from their marginal products, so that:

$$r = \frac{f_k}{\mu} \tag{2}$$

$$w = \frac{f_n}{\mu} \tag{3}$$

The household's Hamiltonian is:

$$H : e^{-\rho t} \frac{c^{1-\gamma} e^{(1-\gamma)v(n)}}{1-\gamma} + \lambda \left[(1-\tau_k) rk + (1-\tau_n) wn + (1-\tau_\pi) \pi - c + T \right]$$

where π is pre-tax profit for the firms, equal to the fraction θ of operating profit that is not dissipated by costs of entry. Households take their share of this profit as exogenous to their actions; that is, they consider investment of another unit of capital as yielding the after tax rate of return $(1-\tau_k) r$. Taking first-order conditions of H, we get:

$$\begin{aligned} FOC_c &: \; e^{-\rho t} c^{-\gamma} e^{(1-\gamma)v(n)} = \lambda \\ FOC_n &: \; e^{-\rho t} v'(n) c^{1-\gamma} e^{(1-\gamma)v(n)} = -\lambda (1-\tau_n) w \\ FOC_k &: \; -\dot{\lambda} = \lambda (1-\tau_k) r \end{aligned}$$

Combining the first two FOCs gives result (4);

$$v'(n) c = -(1-\tau_n) w \tag{4}$$

Take the time derivative of the first of these results:

$$\frac{dFOC_c}{dt} : -\rho\lambda - \gamma \frac{\dot{c}}{c} \lambda = \dot{\lambda}$$

In the steady state, this yields result (5):

$$(1-\tau_k) r = \rho + \gamma g \tag{5}$$

In the steady state, total consumption is equal to final output in terms of final goods consumption, non-dissipated profits, less investment. We can use the dynamic budget constraint and set $\dot{k} = gk$ and T equal to total tax revenue to derive (6):

$$c = \frac{f(k,n)}{\mu} + \pi - gk \tag{6}$$

Total tax revenue is:

$$R = \tau_k r k + \tau_n w n + \tau_\pi \pi \tag{7}$$

and finally, non-dissipated profit (pure rents) is:

$$\begin{aligned} \pi &= \theta\left(f - \frac{f}{\mu}\right) \\ \pi &= \theta\frac{\mu-1}{\mu} f(k,n) \end{aligned} \tag{8}$$

These equations (1)-(8) give our full model. We can proceed by first simplifying them to four key results:

$$(1-\tau_k) f_k = \mu(\rho + \gamma g) \tag{9}$$

$$\begin{aligned} v'(n)\left(\frac{f(k,n)}{\mu} + \theta\frac{\mu-1}{\mu} f(k,n) - gk\right) &= -(1-\tau_n)\frac{f_n}{\mu} \\ v'(n)\left[(1+\theta(\mu-1)) f(k,n) - \mu g k\right] &= -(1-\tau_n) f_n \end{aligned} \tag{10}$$

$$R = \frac{1}{\mu}\left[\tau_k f_k k + \tau_n f_n n + \tau_\pi \theta(\mu-1) f(k,n)\right] \tag{11}$$

and,

$$f(k,n) = k^\alpha n^{1-\alpha} - \delta k. \tag{12}$$

10.0.1 Capital tax results

Now, take the total derivative of (11)

$$dR = \frac{1}{\mu}\left[\begin{array}{c} d\tau_k(f_k k) + d\tau_n(f_n n) + d\tau_\pi(\theta(\mu-1) f(k,n)) \\ +dk\left[\tau_k(f_{kk}k + f_k) + \tau_n(f_{nk}n) + \tau_\pi(\theta(\mu-1) f_k)\right] \\ +dn\left[\tau_k(f_{kn}k) + \tau_n(f_{nn}n + f_n) + \tau_\pi(\theta(\mu-1) f_n)\right] \end{array}\right]$$

$$\frac{dR}{d\tau_k} = \frac{1}{\mu}\left[\begin{array}{c} f_k k \\ +\frac{dk}{d\tau_k}\left[\tau_k(f_{kk}k + f_k) + \tau_n(f_{nk}n) + \tau_\pi(\theta(\mu-1) f_k)\right] \\ +\frac{dn}{d\tau_k}\left[\tau_k(f_{kn}k) + \tau_n(f_{nn}n + f_n) + \tau_\pi(\theta(\mu-1) f_n)\right] \end{array}\right]$$

Using Cobb-Douglas and simplifying,

$$\frac{dR}{d\tau_k} = \frac{1}{\mu}\left[\begin{array}{c} \alpha k^\alpha n^{1-\alpha} \\ +\alpha k^\alpha n^{1-\alpha}\left[\tau_k(\alpha) + \tau_n(1-\alpha) + \tau_\pi(\theta(\mu-1))\right]\frac{1}{k}\frac{dk}{d\tau_k} \\ +\alpha k^\alpha n^{1-\alpha}\left[\tau_k(\alpha) + \tau_n(-\alpha+1) + \tau_\pi(\theta(\mu-1))\right]\frac{1-\alpha}{\alpha}\frac{1}{n}\frac{dn}{d\tau_k} \end{array}\right]$$

$$\frac{dR}{d\tau_k} = \frac{\alpha k^{\alpha} n^{1-\alpha}}{\mu} \left\{ 1 + \left[\alpha \tau_k + (1-\alpha)\, \tau_n + \theta\, (\mu - 1)\, \tau_\pi \right] \left[\frac{1}{k} \frac{dk}{d\tau_k} + \frac{1-\alpha}{\alpha} \frac{1}{n} \frac{dn}{d\tau_k} \right] \right\}$$

Note that the static scoring estimate would be:

$$\left.\frac{dR}{d\tau_k}\right|_{static} = \frac{\alpha k^{\alpha} n^{1-\alpha}}{\mu}$$

So, now we need to know $\frac{dk}{d\tau_k}$ and $\frac{dn}{d\tau_k}$. Use (9):

$$(1-\tau_k)\, f_k - \mu\, (\rho + \gamma g) = 0$$

Then,

$$\frac{dk}{d\tau_k} = \frac{f_k - (1-\tau_k)\, f_{kn} \frac{dn}{d\tau_k}}{(1-\tau_k)\, f_{kk}}$$

Use (10) for $\frac{dn}{d\tau_k}$:

$$\begin{aligned} v'(n) &= \frac{-(1-\tau_n)\, f_n}{(1+\theta\,(\mu-1))\, f(k,n) - \mu g k} \\ v'(n) &= \frac{-(1-\tau_n)\,(1-\alpha)\, k^{\alpha} n^{-\alpha}}{(1+\theta\,(\mu-1))\, k^{\alpha} n^{1-\alpha} - \mu g k} \end{aligned}$$

Now, let $z = \frac{k}{n}$, which by (9) is

$$z = \left(\frac{\alpha\,(1-\tau_k)}{\mu\,(\rho + \gamma g)} \right)^{\frac{1}{1-\alpha}}$$

Then,

$$\begin{aligned} v'(n) &= \frac{-(1-\tau_n)\,(1-\alpha)\, z^{\alpha}}{(1+\theta\,(\mu-1))\, z^{\alpha} n - \mu g z n} \\ n v'(n) &= \frac{-(1-\tau_n)\,(1-\alpha)}{(1+\theta\,(\mu-1)) - \mu g z^{1-\alpha}} \\ n v'(n) &= \frac{-(1-\tau_n)\,(1-\alpha)}{(1+\theta\,(\mu-1)) - \mu g \frac{\alpha(1-\tau_k)}{\mu(\rho+\gamma g)}} \\ n v'(n) &= \frac{-(1-\tau_n)\,(1-\alpha)\,(\rho+\gamma g)}{(1+\theta\,(\mu-1))\,(\rho+\gamma g) - \alpha g\,(1-\tau_k)} \end{aligned}$$

Thus,

$$\frac{dn}{d\tau_k} = \frac{(1-\tau_n)\,(1-\alpha)\,(\rho+\gamma g)\,(\alpha g)}{\left[(1+\theta\,(\mu-1))\,(\rho+\gamma g) - \alpha g\,(1-\tau_k)\right]^2} \frac{1}{n v'' + v'}$$

Now, note that, from (10),

$$v'(n)\, c + (1-\tau_n)\, w = 0$$

so

$$\left.\frac{dn}{dw}\right|_{\bar{c}}\frac{w}{n}=\frac{-\left(1-\tau_n\right)}{cv''}\frac{-v'\left(n\right)c}{\left(1-\tau_n\right)n}=\frac{v'\left(n\right)}{nv''}=\sigma$$

where σ is the constant-consumption elasticity of labor supply. Then,

$$\frac{1}{nv''+v'}=\frac{1}{v'}\frac{\sigma}{1+\sigma}=\frac{\left[\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)\right]n}{-\left(1-\tau_n\right)\left(1-\alpha\right)\left(\rho+\gamma g\right)}\frac{\sigma}{1+\sigma}$$

so,

$$\begin{aligned}
\frac{dn}{d\tau_k} &= \left[\begin{array}{c}\frac{(1-\tau_n)(1-\alpha)(\rho+\gamma g)(\alpha g)}{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)]^2}\\ \cdot\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)]n}{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)}\frac{\sigma}{1+\sigma}\end{array}\right]\\
\frac{dn}{d\tau_k} &= \frac{-\alpha g n\sigma}{\left[\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)\right]\left(1+\sigma\right)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{dk}{d\tau_k} &= \frac{\alpha k^{\alpha-1}n^{1-\alpha}-\alpha\left(1-\alpha\right)\left(1-\tau_k\right)k^{\alpha-1}n^{-\alpha}\frac{dn}{d\tau_k}}{-\alpha\left(1-\alpha\right)\left(1-\tau_k\right)k^{\alpha-2}n^{1-\alpha}}\\
\frac{dk}{d\tau_k} &= \frac{-k\left[1-\left(1-\alpha\right)\left(1-\tau_k\right)\frac{1}{n}\frac{dn}{d\tau_k}\right]}{\left(1-\alpha\right)\left(1-\tau_k\right)}\\
\frac{dk}{d\tau_k} &= \frac{-k\left[1-\left(1-\alpha\right)\left(1-\tau_k\right)\frac{-\alpha g\sigma}{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right]}{\left(1-\alpha\right)\left(1-\tau_k\right)}\\
\frac{dk}{d\tau_k} &= -k\frac{\left[\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)\right]\left(1+\sigma\right)+\left(1-\alpha\right)\left(1-\tau_k\right)\alpha g\sigma}{\left(1-\alpha\right)\left(1-\tau_k\right)\left[\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)\right]\left(1+\sigma\right)}
\end{aligned}$$

This is ugly, but plugging it into our equation for the dynamic change in revenue:

$$\frac{dR}{d\tau_k}=\frac{\alpha k^\alpha n^{1-\alpha}}{\mu}\left\{\begin{array}{c}1-\left[\alpha\tau_k+\left(1-\alpha\right)\tau_n+\theta\left(\mu-1\right)\tau_\pi\right]\\ \left[\begin{array}{c}\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)\alpha g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\\ +\frac{1-\alpha}{\alpha}\frac{\alpha g\sigma}{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\end{array}\right]\end{array}\right\}$$

$$\frac{dR}{d\tau_k}=\frac{\alpha k^\alpha n^{1-\alpha}}{\mu}\left\{\begin{array}{c}1-\left[\alpha\tau_k+\left(1-\alpha\right)\tau_n+\theta\left(\mu-1\right)\tau_\pi\right]\\ \left[\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)\alpha g\sigma+(1-\alpha)(1-\alpha)(1-\tau_k)g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right]\end{array}\right\}$$

$$\frac{dR}{d\tau_k}=\frac{\alpha k^\alpha n^{1-\alpha}}{\mu}\left\{\begin{array}{c}1-\left[\alpha\tau_k+\left(1-\alpha\right)\tau_n+\theta\left(\mu-1\right)\tau_\pi\right]\\ \left[\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right]\end{array}\right\}$$

$$\frac{dR}{d\tau_k}=\frac{\alpha k^\alpha n^{1-\alpha}}{\mu}\left\{1-\frac{\alpha\tau_k+\left(1-\alpha\right)\tau_n+\theta\left(\mu-1\right)\tau_\pi}{\left(1-\alpha\right)\left(1-\tau_k\right)}-\frac{\alpha\tau_k+\left(1-\alpha\right)\tau_n+\theta\left(\mu-1\right)\tau_\pi}{\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)}\frac{\sigma}{1+\sigma}g\right\}$$

Now, this expression is very similar to our expression for a model without markups. If $\mu = 1$, it collapses to that model, as expected. Note, however, that it also collapses to the no-markup example if $\theta = 0$, that is if all pure profits are taken up by entry costs. Calculate the term inside the brackets for our standard parameter values $\alpha = \frac{1}{3}, \tau_k = \tau_n = \frac{1}{4}, \rho = .05, g = .02, \sigma = \frac{1}{2}, \gamma = 1$ and a few illustrative cases:

For $\mu = \frac{5}{4}, \theta = 1, \tau_\pi = \frac{1}{4}$

$1 - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1\left(\frac{5}{4}-1\right)\frac{1}{4}}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1\left(\frac{5}{4}-1\right)\frac{1}{4}}{\left(1+1\left(\frac{5}{4}-1\right)\right)(.05+.02) - \frac{1}{3}(.02)\left(1-\frac{1}{4}\right)} \frac{\frac{1}{2}}{1+\frac{1}{2}} (.02) = 0.349\,75$

For $\mu = 1, \theta = 1, , \tau_\pi = \frac{1}{4}$, no-markup case

$1 - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1(1-1)\frac{1}{4}}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1(1-1)\frac{1}{4}}{(1+1(1-1))(.05+.02) - \frac{1}{3}(.02)\left(1-\frac{1}{4}\right)} \frac{\frac{1}{2}}{1+\frac{1}{2}} (.02) = 0.474\,36$

For $\mu = \frac{5}{4}, \theta = 0, \tau_\pi = \frac{1}{4}$, fully dissipated rents

$1 - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 0\left(\frac{5}{4}-1\right)\frac{1}{4}}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 0\left(\frac{5}{4}-1\right)\frac{1}{4}}{\left(1+0\left(\frac{5}{4}-1\right)\right)(.05+.02) - \frac{1}{3}(.02)\left(1-\frac{1}{4}\right)} \frac{\frac{1}{2}}{1+\frac{1}{2}} (.02) = 0.474\,36$

For $\mu = \frac{11}{10}, \theta = 1, \tau_\pi = \frac{1}{4}$, a smaller markup

$1 - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1\left(\frac{11}{10}-1\right)\frac{1}{4}}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1\left(\frac{11}{10}-1\right)\frac{1}{4}}{\left(1+1\left(\frac{11}{10}-1\right)\right)(.05+.02) - \frac{1}{3}(.02)\left(1-\frac{1}{4}\right)} \frac{\frac{1}{2}}{1+\frac{1}{2}} (.02) = 0.424\,54$

For $\mu = \frac{5}{4}, \theta = \frac{1}{2}, \tau_\pi = \frac{1}{4}$, half dissipation

$1 - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + \frac{1}{2}\left(\frac{5}{4}-1\right)\frac{1}{4}}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + \frac{1}{2}\left(\frac{5}{4}-1\right)\frac{1}{4}}{\left(1+\frac{1}{2}\left(\frac{5}{4}-1\right)\right)(.05+.02) - \frac{1}{3}(.02)\left(1-\frac{1}{4}\right)} \frac{\frac{1}{2}}{1+\frac{1}{2}} (.02) = 0.412\,08$

For $\mu = \frac{5}{4}, \theta = 1, \tau_\pi = 0$, profits untaxed but markups not dissipated

$1 - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1\left(\frac{5}{4}-1\right)0}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} - \frac{\frac{1}{3}\frac{1}{4} + \left(1-\frac{1}{3}\right)\frac{1}{4} + 1\left(\frac{5}{4}-1\right)0}{\left(1+1\left(\frac{5}{4}-1\right)\right)(.05+.02) - \frac{1}{3}(.02)\left(1-\frac{1}{4}\right)} \frac{\frac{1}{2}}{1+\frac{1}{2}} (.02) = 0.479\,80$

Note, by the way, that this is the result from our original paper, assuming profits are not taxed and markups fully non-dissipated: that is, we had

$1 - \frac{\alpha\tau_k + (1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)} - \frac{\alpha\tau_k + (1-\alpha)\tau_n}{\mu(\rho+\gamma g) - \alpha g(1-\tau_k)} \frac{\sigma}{1+\sigma} g = 1 - \frac{\alpha\tau_k + (1-\alpha)\tau_n + \theta(\mu-1)0}{(1-\alpha)(1-\tau_k)} - \frac{\alpha\tau_k + (1-\alpha)\tau_n + \theta(\mu-1)0}{(1+1(\mu-1))(\rho+\gamma g) - \alpha g(1-\tau_k)} \frac{\sigma}{1+\sigma} g$

But, this is probably an unrealistic assumption. Profits are taxed and are probably partially dissipated. The case of $\mu = \frac{5}{4}, \theta = \frac{1}{2}, \tau_\pi = \frac{1}{4}$ may provide a good benchmark, and it indicates that the markup raises the feedback effect to 59% (from 53% in the no-markup case).

10.0.2 Labor tax results

$$dR = \frac{1}{\mu} \begin{bmatrix} d\tau_k (f_k k) + d\tau_n (f_n n) + d\tau_\pi (\theta (\mu - 1) f (k, n)) \\ +dk [\tau_k (f_{kk} k + f_k) + \tau_n (f_{nk} n) + \tau_\pi (\theta (\mu - 1) f_k)] \\ +dn [\tau_k (f_{kn} k) + \tau_n (f_{nn} n + f_n) + \tau_\pi (\theta (\mu - 1) f_n)] \end{bmatrix}$$

$$\frac{dR}{d\tau_n} = \frac{1}{\mu} \begin{bmatrix} f_n n \\ +\frac{dk}{d\tau_n} [\tau_k (f_{kk} k + f_k) + \tau_n (f_{nk} n) + \tau_\pi (\theta (\mu - 1) f_k)] \\ +\frac{dn}{d\tau_n} [\tau_k (f_{kn} k) + \tau_n (f_{nn} n + f_n) + \tau_\pi (\theta (\mu - 1) f_n)] \end{bmatrix}$$

For $\frac{dk}{d\tau_n}$, use result (9):

$$(1 - \tau_k) f_k - \mu (\rho + \gamma g) = 0 \tag{9}$$

$$
\begin{aligned}
\frac{dk}{d\tau_n} &= -\frac{(1-\tau_k) f_{kn} \frac{dn}{d\tau_n}}{(1-\tau_k) f_{kk}} \\
\frac{dk}{d\tau_n} &= -\frac{f_{kn}}{f_{kk}} \frac{dn}{d\tau_n} \\
\frac{dk}{d\tau_n} &= -\frac{\alpha(1-\alpha) k^{\alpha-1} n^{-\alpha}}{\alpha(\alpha-1) k^{\alpha-2} n^{1-\alpha}} \frac{dn}{d\tau_n} \\
\frac{dk}{d\tau_n} &= \frac{k}{n} \frac{dn}{d\tau_n}
\end{aligned}
$$

For $\frac{dn}{d\tau_n}$, use results from the earlier analysis:

$$
nv'(n) + \frac{(1-\tau_n)(1-\alpha)(\rho+\gamma g)}{(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)} = 0
$$

$$
\begin{aligned}
\frac{dn}{d\tau_n} &= \frac{(1-\alpha)(\rho+\gamma g)}{(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)} \frac{1}{nv'' + v'} \\
\frac{dn}{d\tau_n} &= \frac{(1-\alpha)(\rho+\gamma g)}{(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)} \frac{[(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)]\, n}{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)} \frac{\sigma}{1+\sigma} \\
\frac{dn}{d\tau_n} &= \frac{-n}{(1-\tau_n)} \frac{\sigma}{1+\sigma}
\end{aligned}
$$

Therefore,

$$
\frac{dk}{d\tau_n} = \frac{-k}{(1-\tau_n)} \frac{\sigma}{1+\sigma}
$$

So, using Cobb-Douglas and simplifying,

$$
\frac{dR}{d\tau_n} = \frac{1}{\mu} \left[\begin{array}{c} (1-\alpha) k^{\alpha} n^{1-\alpha} \\ +(1-\alpha) k^{\alpha} n^{1-\alpha} \left[\tau_k(\alpha) + \tau_n(1-\alpha) + \tau_\pi(\theta(\mu-1))\right] \frac{\alpha}{1-\alpha} \frac{1}{k} \frac{dk}{d\tau_k} \\ +(1-\alpha) k^{\alpha} n^{1-\alpha} \left[\tau_k(\alpha) + \tau_n(-\alpha+1) + \tau_\pi(\theta(\mu-1))\right] \frac{1}{n} \frac{dn}{d\tau_k} \end{array} \right]
$$

$$
\frac{dR}{d\tau_n} = \frac{(1-\alpha) k^{\alpha} n^{1-\alpha}}{\mu} \left\{ 1 + \left[\alpha\tau_k + (1-\alpha)\tau_n + \theta(\mu-1)\tau_\pi\right] \left[\frac{\alpha}{1-\alpha} \frac{1}{k} \frac{dk}{d\tau_k} + \frac{1}{n} \frac{dn}{d\tau_k} \right] \right\}
$$

$$
\frac{dR}{d\tau_n} = \frac{(1-\alpha) k^{\alpha} n^{1-\alpha}}{\mu} \left\{ 1 + \left[\alpha\tau_k + (1-\alpha)\tau_n + \theta(\mu-1)\tau_\pi\right] \left[\frac{\alpha}{1-\alpha} \frac{1}{k} \frac{-k}{(1-\tau_n)} \frac{\sigma}{1+\sigma} + \frac{1}{n} \frac{-n}{(1-\tau_n)} \frac{\sigma}{1+\sigma} \right] \right\}
$$

$$
\frac{dR}{d\tau_n} = \frac{(1-\alpha) k^{\alpha} n^{1-\alpha}}{\mu} \left\{ 1 - \left[\alpha\tau_k + (1-\alpha)\tau_n + \theta(\mu-1)\tau_\pi\right] \left[\frac{\alpha}{(1-\alpha)(1-\tau_n)} \frac{\sigma}{1+\sigma} + \frac{1-\alpha}{(1-\alpha)(1-\tau_n)} \frac{\sigma}{1+\sigma} \right.\right.
$$

$$
\frac{dR}{d\tau_n} = \frac{(1-\alpha) k^{\alpha} n^{1-\alpha}}{\mu} \left\{ 1 - \frac{\alpha\tau_k + (1-\alpha)\tau_n + \theta(\mu-1)\tau_\pi}{(1-\alpha)(1-\tau_n)} \frac{\sigma}{1+\sigma} \right\}
$$

10.1 When profits are taxed at same rate as capital

On the production side, note that

$$f(k,n) = k^{\alpha} n^{1-\alpha}. \tag{1}$$

However, due to imperfect competition, factor returns are distorted away from their marginal products, so that:

$$r = \frac{f_k}{\mu} \tag{2}$$

$$w = \frac{f_n}{\mu} \tag{3}$$

The household's Hamiltonian is:

$$H : e^{-\rho t} \frac{c^{1-\gamma} e^{(1-\gamma)v(n)}}{1-\gamma} + \lambda \left[(1-\tau_k) rk + (1-\tau_n) wn + (1-\tau_k) \pi - c + T \right]$$

where π is pre-tax profit for the firms, equal to the fraction θ of operating profit that is not dissipated by costs of entry. Households pay the same rate of tax on profits as they do on capital income, though they take their share of profits as exogenous and assume that another unit of capital investment will pay back $(1-\tau_k)r$. Taking first-order conditions of H, we get:

$$\begin{aligned} FOC_c &: \quad e^{-\rho t} c^{-\gamma} e^{(1-\gamma)v(n)} = \lambda \\ FOC_n &: \quad e^{-\rho t} v'(n) c^{1-\gamma} e^{(1-\gamma)v(n)} = -\lambda (1-\tau_n) w \\ FOC_k &: \quad -\dot{\lambda} = \lambda (1-\tau_k) r \end{aligned}$$

Combining the first two FOCs gives result (4);

$$v'(n)\, c = -(1-\tau_n) w \tag{4}$$

Take the time derivative of the first of these results:

$$\frac{dFOC_c}{dt} : -\rho\lambda - \gamma \frac{\dot{c}}{c} \lambda = \dot{\lambda}$$

In the steady state, this yields result (5):

$$(1-\tau_k) r = \rho + \gamma g \tag{5}$$

In the steady state, total consumption is equal to final output in terms of final goods consumption, non-dissipated profits, less investment. We can use the dynamic budget constraint and set $\dot{k} = gk$ and T equal to total tax revenue to derive (6):

$$c = \frac{f(k,n)}{\mu} + \pi - gk \tag{6}$$

Total tax revenue is:

$$R = \tau_k (rk + \pi) + \tau_n wn \tag{7}$$

and finally, non-dissipated profit (pure rents) is:

$$\begin{aligned} \pi &= \theta \left(f - \frac{f}{\mu} \right) \\ \pi &= \theta \frac{\mu - 1}{\mu} f(k, n) \end{aligned} \tag{8}$$

These equations (1)-(8) give our full model. We can proceed by first simplifying them to four key results:

$$(1 - \tau_k) f_k = \mu (\rho + \gamma g) \tag{9}$$

$$\begin{aligned} v'(n) \left(\frac{f(k,n)}{\mu} + \theta \frac{\mu - 1}{\mu} f(k,n) - gk \right) &= -(1 - \tau_n) \frac{f_n}{\mu} \\ v'(n) \left[(1 + \theta (\mu - 1)) f(k,n) - \mu g k \right] &= -(1 - \tau_n) f_n \end{aligned} \tag{10}$$

$$R = \frac{1}{\mu} \left[\tau_k f_k k + \tau_n f_n n + \tau_k \theta (\mu - 1) f(k,n) \right] \tag{11}$$

and,

$$f(k,n) = k^{\alpha} n^{1-\alpha}. \tag{12}$$

10.1.1 Capital tax results

Now, take the total derivative of (11)

$$dR = \frac{1}{\mu} \left[\begin{array}{c} d\tau_k (f_k k + \theta (\mu - 1) f(k,n)) + d\tau_n (f_n n) \\ +dk \left[\tau_k (f_{kk} k + f_k) + \tau_n (f_{nk} n) + \tau_k (\theta (\mu - 1) f_k) \right] \\ +dn \left[\tau_k (f_{kn} k) + \tau_n (f_{nn} n + f_n) + \tau_k (\theta (\mu - 1) f_n) \right] \end{array} \right]$$

$$\frac{dR}{d\tau_k} = \frac{1}{\mu} \left[\begin{array}{c} f_k k + \theta (\mu - 1) f(k,n) \\ +\frac{dk}{d\tau_k} \left[\tau_k (f_{kk} k + f_k) + \tau_n (f_{nk} n) + \tau_k (\theta (\mu - 1) f_k) \right] \\ +\frac{dn}{d\tau_k} \left[\tau_k (f_{kn} k) + \tau_n (f_{nn} n + f_n) + \tau_k (\theta (\mu - 1) f_n) \right] \end{array} \right]$$

Using Cobb-Douglas and simplifying,

$$\frac{dR}{d\tau_k} = \frac{1}{\mu} \left[\begin{array}{c} (\alpha + \theta (\mu - 1)) k^{\alpha} n^{1-\alpha} \\ +\alpha k^{\alpha} n^{1-\alpha} \left[\tau_k (\alpha) + \tau_n (1 - \alpha) + \tau_k (\theta (\mu - 1)) \right] \frac{1}{k} \frac{dk}{d\tau_k} \\ +\alpha k^{\alpha} n^{1-\alpha} \left[\tau_k (\alpha) + \tau_n (-\alpha + 1) + \tau_k (\theta (\mu - 1)) \right] \frac{1-\alpha}{\alpha} \frac{1}{n} \frac{dn}{d\tau_k} \end{array} \right]$$

$$\frac{dR}{d\tau_k} = \frac{1}{\mu} \left[\begin{array}{c} (\alpha + \theta (\mu - 1)) k^{\alpha} n^{1-\alpha} \\ +\alpha k^{\alpha} n^{1-\alpha} \left[(\alpha + \theta (\mu - 1)) \tau_k + \tau_n (1 - \alpha) \right] \frac{1}{k} \frac{dk}{d\tau_k} \\ +\alpha k^{\alpha} n^{1-\alpha} \left[(\alpha + \theta (\mu - 1)) \tau_k + \tau_n (1 - \alpha) \right] \frac{1-\alpha}{\alpha} \frac{1}{n} \frac{dn}{d\tau_k} \end{array} \right]$$

$$\frac{dR}{d\tau_k} = (\alpha + \theta(\mu - 1)) \frac{1}{\mu} k^\alpha n^{1-\alpha} \begin{bmatrix} 1 \\ +\alpha \left[\tau_k + \tau_n \frac{(1-\alpha)}{(\alpha+\theta(\mu-1))}\right] \frac{1}{k}\frac{dk}{d\tau_k} \\ +\alpha \left[\tau_k + \tau_n \frac{(1-\alpha)}{(\alpha+\theta(\mu-1))}\right] \frac{1-\alpha}{\alpha}\frac{1}{n}\frac{dn}{d\tau_k} \end{bmatrix}$$

$$\frac{dR}{d\tau_k} = \frac{(\alpha + \theta(\mu - 1))}{\mu} k^\alpha n^{1-\alpha} \begin{bmatrix} 1 \\ +\frac{\alpha}{(\alpha+\theta(\mu-1))}\left[(\alpha+\theta(\mu-1))\tau_k + (1-\alpha)\tau_n\right]\frac{1}{k}\frac{dk}{d\tau_k} \\ +\frac{\alpha}{(\alpha+\theta(\mu-1))}\left[(\alpha+\theta(\mu-1))\tau_k + (1-\alpha)\tau_n\right]\frac{1-\alpha}{\alpha}\frac{1}{n}\frac{dn}{d\tau_k} \end{bmatrix}$$

Call $\tilde{\alpha} \equiv (\alpha + \theta(\mu - 1))$, the adjusted capital share in this model with markups. Then,

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu} k^\alpha n^{1-\alpha} \left[1 + \frac{\alpha}{\tilde{\alpha}}\left[\tilde{\alpha}\tau_k + (1-\alpha)\tau_n\right]\frac{1}{k}\frac{dk}{d\tau_k} + \frac{\alpha}{\tilde{\alpha}}\left[\tilde{\alpha}\tau_k + (1-\alpha)\tau_n\right]\frac{1-\alpha}{\alpha}\frac{1}{n}\frac{dn}{d\tau_k}\right]$$

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu} k^\alpha n^{1-\alpha} \left[1 + \frac{\alpha}{\tilde{\alpha}}\left[\tilde{\alpha}\tau_k + (1-\alpha)\tau_n\right]\left[\frac{1}{k}\frac{dk}{d\tau_k} + \frac{1-\alpha}{\alpha}\frac{1}{n}\frac{dn}{d\tau_k}\right]\right]$$

Note that the static scoring estimate would be:

$$\left.\frac{dR}{d\tau_k}\right|_{static} = \left(rk + \theta\frac{\mu-1}{\mu} f(k,n)\right) \quad (7)$$

$$\left.\frac{dR}{d\tau_k}\right|_{static} = \left(\frac{f_k}{\mu}k + \theta\frac{\mu-1}{\mu} f(k,n)\right) \quad (12)$$

$$\left.\frac{dR}{d\tau_k}\right|_{static} = \frac{1}{\mu}\left(\alpha k^\alpha n^{1-\alpha} + \theta(\mu-1) k^\alpha n^{1-\alpha}\right) \quad (13)$$

$$\left.\frac{dR}{d\tau_k}\right|_{static} = \frac{\tilde{\alpha}}{\mu} k^\alpha n^{1-\alpha}$$

So, now we need to know $\frac{dk}{d\tau_k}$ and $\frac{dn}{d\tau_k}$. Use (9):

$$(1-\tau_k) f_k - \mu(\rho + \gamma g) = 0$$

Then,

$$\frac{dk}{d\tau_k} = \frac{f_k - (1-\tau_k) f_{kn}\frac{dn}{d\tau_k}}{(1-\tau_k) f_{kk}}$$

Use (10) for $\frac{dn}{d\tau_k}$:

$$v'(n) = \frac{-(1-\tau_n) f_n}{(1+\theta(\mu-1)) f(k,n) - \mu g k}$$

$$v'(n) = \frac{-(1-\tau_n)(1-\alpha) k^\alpha n^{-\alpha}}{(1+\theta(\mu-1)) k^\alpha n^{1-\alpha} - \mu g k}$$

Now, let $z = \frac{k}{n}$, which by (9) is

$$z = \left(\frac{\alpha (1 - \tau_k)}{\mu (\rho + \gamma g)} \right)^{\frac{1}{1-\alpha}}$$

Then,

$$\begin{aligned}
v'(n) &= \frac{-(1-\tau_n)(1-\alpha) z^{\alpha}}{(1+\theta(\mu-1)) z^{\alpha} n - \mu g z n} \\
n v'(n) &= \frac{-(1-\tau_n)(1-\alpha)}{(1+\theta(\mu-1)) - \mu g z^{1-\alpha}} \\
n v'(n) &= \frac{-(1-\tau_n)(1-\alpha)}{(1+\theta(\mu-1)) - \mu g \frac{\alpha(1-\tau_k)}{\mu(\rho+\gamma g)}} \\
n v'(n) &= \frac{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)}{(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)}
\end{aligned}$$

Thus,

$$\frac{dn}{d\tau_k} = \frac{(1-\tau_n)(1-\alpha)(\rho+\gamma g)(\alpha g)}{[(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)]^2} \frac{1}{n v'' + v'}$$

Now, note that, from (10),

$$v'(n) c + (1-\tau_n) w = 0$$

so

$$\left. \frac{dn}{dw} \right|_{\bar{c}} \frac{w}{n} = \frac{-(1-\tau_n)}{c v''} \frac{-v'(n) c}{(1-\tau_n) n} = \frac{v'(n)}{n v''} = \sigma$$

where σ is the constant-consumption elasticity of labor supply. Then,

$$\frac{1}{n v'' + v'} = \frac{1}{v'} \frac{\sigma}{1+\sigma} = \frac{[(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)] n}{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)} \frac{\sigma}{1+\sigma}$$

so,

$$\begin{aligned}
\frac{dn}{d\tau_k} &= \left[\begin{array}{c} \frac{(1-\tau_n)(1-\alpha)(\rho+\gamma g)(\alpha g)}{[(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)]^2} \\ \cdot \frac{[(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)] n}{-(1-\tau_n)(1-\alpha)(\rho+\gamma g)} \frac{\sigma}{1+\sigma} \end{array} \right] \\
\frac{dn}{d\tau_k} &= \frac{-\alpha g n \sigma}{[(1+\theta(\mu-1))(\rho+\gamma g) - \alpha g (1-\tau_k)](1+\sigma)}
\end{aligned}$$

Thus,

$$\frac{dk}{d\tau_k} = \frac{\alpha k^{\alpha-1}n^{1-\alpha} - \alpha(1-\alpha)(1-\tau_k)k^{\alpha-1}n^{-\alpha}\frac{dn}{d\tau_k}}{-\alpha(1-\alpha)(1-\tau_k)k^{\alpha-2}n^{1-\alpha}}$$

$$\frac{dk}{d\tau_k} = \frac{-k\left[1-(1-\alpha)(1-\tau_k)\frac{1}{n}\frac{dn}{d\tau_k}\right]}{(1-\alpha)(1-\tau_k)}$$

$$\frac{dk}{d\tau_k} = \frac{-k\left[1-(1-\alpha)(1-\tau_k)\frac{-\alpha g\sigma}{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right]}{(1-\alpha)(1-\tau_k)}$$

$$\frac{dk}{d\tau_k} = -k\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)\alpha g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}$$

This is ugly, but plugging it into our equation for the dynamic change in revenue:

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}\left\{1+\frac{\alpha}{\tilde{\alpha}}[\tilde{\alpha}\tau_k+(1-\alpha)\tau_n]\left[\frac{1}{k}\frac{dk}{d\tau_k}+\frac{1-\alpha}{\alpha}\frac{1}{n}\frac{dn}{d\tau_k}\right]\right\}$$

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}\left\{\begin{array}{c} 1-\frac{\alpha}{\tilde{\alpha}}[\tilde{\alpha}\tau_k+(1-\alpha)\tau_n]\cdot \\ \left[\begin{array}{c}\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)\alpha g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)} \\ +\frac{1-\alpha}{\alpha}\frac{\alpha g\sigma}{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\end{array}\right]\end{array}\right\}$$

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}\left\{\begin{array}{c} 1-\frac{\alpha}{\tilde{\alpha}}[\tilde{\alpha}\tau_k+(1-\alpha)\tau_n]\cdot \\ \left[\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)\alpha g\sigma+(1-\alpha)(1-\alpha)(1-\tau_k)g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right]\end{array}\right\}$$

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}\left\{\begin{array}{c} 1-\frac{\alpha}{\tilde{\alpha}}[\tilde{\alpha}\tau_k+(1-\alpha)\tau_n]\cdot \\ \left[\frac{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)+(1-\alpha)(1-\tau_k)g\sigma}{(1-\alpha)(1-\tau_k)[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right]\end{array}\right\}$$

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}\left\{1-\frac{\alpha}{\tilde{\alpha}}[\tilde{\alpha}\tau_k+(1-\alpha)\tau_n]\left[\frac{1}{(1-\alpha)(1-\tau_k)}+\frac{g\sigma}{[(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)](1+\sigma)}\right.\right.$$

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}\left\{1-\frac{\alpha}{\tilde{\alpha}}\left[\frac{\tilde{\alpha}\tau_k+(1-\alpha)\tau_n}{(1-\alpha)(1-\tau_k)}+\frac{\tilde{\alpha}\tau_k+(1-\alpha)\tau_n}{(1+\theta(\mu-1))(\rho+\gamma g)-\alpha g(1-\tau_k)}\frac{\sigma}{1+\sigma}g\right]\right\}$$

Now, this expression is somewhat different from our expression when profits were taxed at a different rate. That expression was derived in the previous section. To see how taxing profits at the same rate as capital affects dynamic scoring, remember that the static scoring estimate of a capital tax cut would be

$$\frac{dR}{d\tau_k} = \frac{\tilde{\alpha}}{\mu}k^{\alpha}n^{1-\alpha}.$$

So, we calculate the term inside the brackets for our standard parameter values $\alpha = \frac{1}{3}, \tau_k = \tau_n = \frac{1}{4}, \rho = .05, g = (.02), \sigma = \frac{1}{2}, \gamma = 1$ and a few illustrative cases:

For $\mu = \frac{5}{4}, \theta = 1, \tau_\pi = \left(\frac{1}{4}\right)$, we first calculate $\tilde{\alpha} \equiv \alpha + \theta(\mu - 1) = \frac{1}{3} + \left(\frac{5}{4} - 1\right) = 0.583\,33$

$$1-\frac{\frac{1}{3}}{0.58333}\left(\frac{0.58333\left(\frac{1}{4}\right)+\left(1-\frac{1}{3}\right)\left(\frac{1}{4}\right)}{\left(1-\frac{1}{3}\right)\left(1-\left(\frac{1}{4}\right)\right)}+\frac{0.58333\left(\frac{1}{4}\right)+\left(1-\frac{1}{3}\right)\left(\frac{1}{4}\right)}{\left(1+1\left(\frac{5}{4}-1\right)\right)\left((.05)+1(.02)\right)-\frac{1}{3}(.02)\left(1-\left(\frac{1}{4}\right)\right)}\frac{\frac{1}{2}}{1+\frac{1}{2}}(.02)\right)=$$

0.62843

So the feedback effect has fallen to only 37 percent, rather than 53 percent, when profits are taxed at the same rate as capital.

10.1.2 Labor tax results

$$dR=\frac{1}{\mu}\left[\begin{array}{c} d\tau_k\left(f_k k\right)+d\tau_n\left(f_n n\right)+d\tau_\pi\left(\theta\left(\mu-1\right)f\left(k,n\right)\right) \\ +dk\left[\tau_k\left(f_{kk}k+f_k\right)+\tau_n\left(f_{nk}n\right)+\tau_\pi\left(\theta\left(\mu-1\right)f_k\right)\right] \\ +dn\left[\tau_k\left(f_{kn}k\right)+\tau_n\left(f_{nn}n+f_n\right)+\tau_\pi\left(\theta\left(\mu-1\right)f_n\right)\right] \end{array}\right]$$

$$\frac{dR}{d\tau_n}=\frac{1}{\mu}\left[\begin{array}{c} f_n n \\ +\frac{dk}{d\tau_n}\left[\tau_k\left(f_{kk}k+f_k\right)+\tau_n\left(f_{nk}n\right)+\tau_\pi\left(\theta\left(\mu-1\right)f_k\right)\right] \\ +\frac{dn}{d\tau_n}\left[\tau_k\left(f_{kn}k\right)+\tau_n\left(f_{nn}n+f_n\right)+\tau_\pi\left(\theta\left(\mu-1\right)f_n\right)\right] \end{array}\right]$$

For $\frac{dk}{d\tau_n}$, use result (9):

$$\left(1-\tau_k\right)f_k-\mu\left(\rho+\gamma g\right)=0 \tag{9}$$

$$\begin{aligned} \frac{dk}{d\tau_n} &= -\frac{(1-\tau_k)f_{kn}\frac{dn}{d\tau_n}}{(1-\tau_k)f_{kk}} \\ \frac{dk}{d\tau_n} &= -\frac{f_{kn}}{f_{kk}}\frac{dn}{d\tau_n} \\ \frac{dk}{d\tau_n} &= -\frac{\alpha(1-\alpha)k^{\alpha-1}n^{-\alpha}}{\alpha(\alpha-1)k^{\alpha-2}n^{1-\alpha}}\frac{dn}{d\tau_n} \\ \frac{dk}{d\tau_n} &= \frac{k}{n}\frac{dn}{d\tau_n} \end{aligned}$$

For $\frac{dn}{d\tau_n}$, use results from the earlier analysis:

$$nv'\left(n\right)+\frac{\left(1-\tau_n\right)\left(1-\alpha\right)\left(\rho+\gamma g\right)}{\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)}=0$$

$$\begin{aligned} \frac{dn}{d\tau_n} &= \frac{\left(1-\alpha\right)\left(\rho+\gamma g\right)}{\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)}\frac{1}{nv''+v'} \\ \frac{dn}{d\tau_n} &= \frac{\left(1-\alpha\right)\left(\rho+\gamma g\right)}{\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)}\frac{\left[\left(1+\theta\left(\mu-1\right)\right)\left(\rho+\gamma g\right)-\alpha g\left(1-\tau_k\right)\right]n}{-\left(1-\tau_n\right)\left(1-\alpha\right)\left(\rho+\gamma g\right)}\frac{\sigma}{1+\sigma} \\ \frac{dn}{d\tau_n} &= \frac{-n}{\left(1-\tau_n\right)}\frac{\sigma}{1+\sigma} \end{aligned}$$

Therefore,

$$\frac{dk}{d\tau_n}=\frac{-k}{\left(1-\tau_n\right)}\frac{\sigma}{1+\sigma}$$

So, using Cobb-Douglas and simplifying,

$$\frac{dR}{d\tau_n}=\frac{1}{\mu}\left[\begin{array}{c}(1-\alpha)k^{\alpha}n^{1-\alpha}\\+(1-\alpha)\,k^{\alpha}n^{1-\alpha}\left[\tau_k\,(\alpha)+\tau_n\,(1-\alpha)+\tau_\pi\,(\theta\,(\mu-1))\right]\frac{\alpha}{1-\alpha}\frac{1}{k}\frac{dk}{d\tau_k}\\+(1-\alpha)\,k^{\alpha}n^{1-\alpha}\left[\tau_k\,(\alpha)+\tau_n\,(-\alpha+1)+\tau_\pi\,(\theta\,(\mu-1))\right]\frac{1}{n}\frac{dn}{d\tau_k}\end{array}\right]$$

$$\frac{dR}{d\tau_n}=\frac{(1-\alpha)\,k^{\alpha}n^{1-\alpha}}{\mu}\left\{1+\left[\alpha\tau_k+(1-\alpha)\,\tau_n+\theta\,(\mu-1)\,\tau_\pi\right]\left[\frac{\alpha}{1-\alpha}\frac{1}{k}\frac{dk}{d\tau_k}+\frac{1}{n}\frac{dn}{d\tau_k}\right]\right\}$$

$$\frac{dR}{d\tau_n}=\frac{(1-\alpha)\,k^{\alpha}n^{1-\alpha}}{\mu}\left\{1+\left[\alpha\tau_k+(1-\alpha)\,\tau_n+\theta\,(\mu-1)\,\tau_\pi\right]\left[\frac{\alpha}{1-\alpha}\frac{1}{k}\frac{-k}{(1-\tau_n)}\frac{\sigma}{1+\sigma}+\frac{1}{n}\frac{-n}{(1-\tau_n)}\frac{\sigma}{1+\sigma}\right]\right\}$$

$$\frac{dR}{d\tau_n}=\frac{(1-\alpha)\,k^{\alpha}n^{1-\alpha}}{\mu}\left\{1-\left[\alpha\tau_k+(1-\alpha)\,\tau_n+\theta\,(\mu-1)\,\tau_\pi\right]\left[\frac{\alpha}{(1-\alpha)\,(1-\tau_n)}\frac{\sigma}{1+\sigma}+\frac{1-\alpha}{(1-\alpha)\,(1-\tau_n)}\frac{\sigma}{1+\sigma}\right.\right.$$

$$\frac{dR}{d\tau_n}=\frac{(1-\alpha)\,k^{\alpha}n^{1-\alpha}}{\mu}\left\{1-\frac{\alpha\tau_k+(1-\alpha)\,\tau_n+\theta\,(\mu-1)\,\tau_\pi}{(1-\alpha)\,(1-\tau_n)}\frac{\sigma}{1+\sigma}\right\}$$

11 Transitional Dynamics

For the transitional dynamics, we derive differential equations that describe the time path of the capital stock, consumption, and the labor supply. Recall that in our model of Section 2, we assume that labor enters the utility function inside $v(n)$. We specify the functional form for simplicity, but would not have to. The equivalent results with no specified functional form are derived at the end of this section.

$$v(n)=-\theta n^{1+\frac{1}{\sigma}}$$

Note that this form implies:

$$\begin{aligned}\sigma &= \frac{v'(N)}{v''(N)\cdot N}\\&=\frac{-\theta\left(1+\frac{1}{\sigma}\right)n^{\frac{1}{\sigma}}}{-\theta\left(1+\frac{1}{\sigma}\right)\frac{1}{\sigma}n^{\frac{1}{\sigma}}}\\\sigma &= \sigma\end{aligned}$$

So σ is our standard constant consumption elasticity of labor supply.

As mentioned in the main text, the nature of our system is that, when taxes are changed, consumption and labor supply are free to jump immediately, while the capital stock is momentarily fixed at its original level. The mathematical difficulty is finding these jump values. To be even more specific, we have three key points in time when we need to know the values of the capital stock, consumption, and labor: before the tax cut, immediately after the tax cut, and the steady state after the tax cut. We will therefore have nine values of these three variables to calculate, which we will denote $c_0, n_0, k_0, c_\varepsilon, n_\varepsilon, k_\varepsilon, c^*, n^*, k^*$. The before-tax values and the steady state values can be derived with the standard steady-state conditions–the tricky values are the three at $t = \varepsilon$. Fortunately, the capital stock is momentarily fixed, so $k_0 = k_\varepsilon$. We can derive the remaining two values, the "jump" values c_ε and n_ε, with the differential equations that describe the transition path to the steady state. The differential equations will give us a way to calculate c, n at any point in time, and specifically at $t = \varepsilon$ as $\varepsilon \to 0$. To find useable expressions of these differential equations, we begin with the household's utility function:

$$H = \begin{bmatrix} e_t^{-\rho t} \frac{\left(c_t e^{gt}\right)^{1-\gamma} e^{(1-\gamma)(-\theta) n^{1+\frac{1}{\sigma}}} - 1}{1-\gamma} + \\ \varphi(t) \left[(1-\tau_n) wN + (1-\tau_k) rk - c - gk + T\right] \end{bmatrix}$$

Performing the household's maximization, we can derive the following results:

$$\frac{\dot{c}_t}{c_t} = \frac{(1-\tau_k)\alpha k^{\alpha-1} N^{1-\alpha} - (1-\gamma)\theta(1+\frac{1}{\sigma})\dot{N} N^{\frac{1}{\sigma}} - \rho - \gamma g)}{\gamma} \quad (33)$$

$$\frac{\dot{N}}{N} = \frac{\rho + (\gamma - 1 + \tau_k)\alpha k^{\alpha-1} N^{1-\alpha} + (1-\alpha)\gamma g - \alpha\gamma\frac{c}{k}}{(\frac{1}{\sigma} + \alpha)\gamma - \theta(1+\frac{1}{\sigma})(1-\gamma)N^{1+\frac{1}{\sigma}}} \quad (34)$$

The firm's decisions are the same as before, so we have the following results. We assume Cobb-Douglas from the start for simplicity.

$$\begin{aligned} \dot{k}_t &= (1-\tau)k_t^\alpha - c_t + \tau k_t^\alpha - g k_t \\ \text{so, } \frac{\dot{k}_t}{k_t} &= k_t^{\alpha-1} N_t^{1-\alpha} - \frac{c_t}{k_t} - g \end{aligned} \quad (35)$$

To derive analytical results, we linearize the system (33)-(35) around the steady state. This will give results that are particularly applicable for small tax cuts. To derive a first-order linear approximation of (33)-(35), rewrite they system in terms of natural logs. We assume that $\gamma = 1$, which is equivalent to log utility, for simplicity.

$$\frac{d \ln c_t}{dt} = \alpha(1-\tau_k) e^{(\alpha-1)(\ln k - \ln n)} - \rho - g \quad (36)$$

$$\frac{d \ln n_t}{dt} = \frac{1}{\alpha + \frac{1}{\sigma}} \left[\alpha \tau_k e^{(\alpha-1)(\ln k - \ln n)} - \alpha e^{(\ln c - \ln k)} + \rho + (1-\alpha) g\right] \quad (37)$$

$$\frac{d \ln k_t}{dt} = e^{(\alpha-1)(\ln k - \ln n)} - e^{(\ln c - \ln k)} - g \quad (38)$$

Now we use the fact that capital, labor, and consumption per efficiency unit are constant in the long run to solve for steady-state expressions in terms of parameters. We will use these later to simplify our first-order approximation of (36)-(38)..

$$
\begin{aligned}
e^{(\alpha-1)(\ln k^*-\ln n^*)} &= \frac{\rho+g}{\alpha(1-\tau_k)} \\
e^{(\ln c^*-\ln k^*)} &= \left(\frac{\rho+g}{\alpha(1-\tau_k)}-g\right)
\end{aligned}
$$

Now we take a first-order Taylor approximation of the log-linear system (36)-(38) around the log deviations of c, n, k from their steady-state values.:

$$
\begin{bmatrix} \frac{d\ln c_t}{dt} \\ \frac{d\ln n_t}{dt} \\ \frac{d\ln k_t}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(\alpha-1)(1-\tau_k)e^{(\alpha-1)(\ln k-\ln n)} & \alpha(\alpha-1)(1-\tau_k)e^{(\alpha-1)(\ln k-\ln n)} \\ \frac{-\alpha e^{(\ln c-\ln k)}}{\alpha+\frac{1}{\sigma}} & \frac{-\alpha(\alpha-1)\tau_k e^{(\alpha-1)(\ln k-\ln n)}}{\alpha+\frac{1}{\sigma}} & \frac{\alpha(\alpha-1)\tau_k e^{(\alpha-1)(\ln k-\ln n)}+\alpha e^{(\ln c-\ln k)}}{\alpha+\frac{1}{\sigma}} \\ -e^{(\ln c-\ln k)} & -(\alpha-1)e^{(\alpha-1)(\ln k-\ln n)} & (\alpha-1)e^{(\alpha-1)(\ln k-\ln n)}+e^{(\ln c-\ln k)} \end{bmatrix}
$$

$$
\begin{bmatrix} \frac{d\ln c_t}{dt} \\ \frac{d\ln n_t}{dt} \\ \frac{d\ln k_t}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(\alpha-1)(1-\tau_k)\frac{\rho+g}{\alpha(1-\tau_k)} & \alpha(\alpha-1)(1-\tau_k)\frac{\rho+g}{\alpha(1-\tau_k)} \\ \frac{-\alpha\left(\frac{\rho+g}{\alpha(1-\tau_k)}-g\right)}{\alpha+\frac{1}{\sigma}} & \frac{-\alpha(\alpha-1)\tau_k\frac{\rho+g}{\alpha(1-\tau_k)}}{\alpha+\frac{1}{\sigma}} & \frac{\alpha(\alpha-1)\tau_k\frac{\rho+g}{\alpha(1-\tau_k)}+\alpha\left(\frac{\rho+g}{\alpha(1-\tau_k)}-g\right)}{\alpha+\frac{1}{\sigma}} \\ -\left(\frac{\rho+g}{\alpha(1-\tau_k)}-g\right) & -(\alpha-1)\frac{\rho+g}{\alpha(1-\tau_k)} & (\alpha-1)\frac{\rho+g}{\alpha(1-\tau_k)}+\left(\frac{\rho+g}{\alpha(1-\tau_k)}-g\right) \end{bmatrix}
$$

$$
\begin{bmatrix} \frac{d\ln c_t}{dt} \\ \frac{d\ln n_t}{dt} \\ \frac{d\ln k_t}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -(\alpha-1)\rho+g & (\alpha-1)\rho+g \\ \frac{-(\rho+g)}{\left(\alpha+\frac{1}{\sigma}\right)(1-\tau_k)}+\frac{\alpha g}{\alpha+\frac{1}{\sigma}} & \frac{-(\alpha-1)\tau_k(\rho+g)}{(1-\tau_k)\left(\alpha+\frac{1}{\sigma}\right)} & \frac{(1+\alpha\tau_k-\tau_k)(\rho+g)}{\left(\alpha+\frac{1}{\sigma}\right)(1-\tau_k)}-\frac{\alpha g}{\alpha+\frac{1}{\sigma}} \\ -\frac{\rho+g}{\alpha(1-\tau_k)}+g & \frac{-(\alpha-1)(\rho+g)}{\alpha(1-\tau_k)} & \frac{\rho+g}{1-\tau_k}-g \end{bmatrix} \begin{bmatrix} \ln(\frac{c_t}{c^*}) \\ \ln(\frac{n_t}{n^*}) \\ \ln(\frac{k_t}{k^*}) \end{bmatrix}
$$

To simplify going forward, we assume $g = 0$. We could have done so earlier, but we have tried to retain generality as long as practical to allow for interested readers to pursue the more general cases that we do not. We will discuss the likely effects of our simplifying assumptions later.

$$
\begin{bmatrix} \frac{d\ln c_t}{dt} \\ \frac{d\ln n_t}{dt} \\ \frac{d\ln k_t}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -(\alpha-1)\rho & (\alpha-1)\rho \\ \frac{-\rho}{\left(\alpha+\frac{1}{\sigma}\right)(1-\tau_k)} & \frac{-(\alpha-1)\tau_k\rho}{(1-\tau_k)\left(\alpha+\frac{1}{\sigma}\right)} & \frac{(1+(\alpha-1)\tau_k)\rho}{\left(\alpha+\frac{1}{\sigma}\right)(1-\tau_k)} \\ -\frac{\rho}{\alpha(1-\tau_k)} & \frac{-(\alpha-1)\rho}{\alpha(1-\tau_k)} & \frac{\rho}{1-\tau_k} \end{bmatrix} \begin{bmatrix} \ln(\frac{c_t}{c^*}) \\ \ln(\frac{n_t}{n^*}) \\ \ln(\frac{k_t}{k^*}) \end{bmatrix}
$$

Call this matrix A. The theory of differential equations tells us that we can use A to derive the transition paths of our variables to their steady-state levels. Specifically, we can find the eigenvalues and eigenvectors associated with A. To do so, we form the characteristic equation of A and find the values of β for which $\det[A-\beta I]=0$. The characteristic equation can be simplified to:

$$
\left[(\alpha+\frac{1}{\sigma})(1-\tau_k)^2\right]\beta^3-\left[(\alpha+\frac{1}{\sigma}+(1-\alpha)\tau_k)(1-\tau_k)\rho\right]\beta^2-\left[\frac{1}{\alpha}((1-\alpha)^2+(\alpha+\frac{1}{\sigma})(1-\alpha))(1-\tau_k)\rho^2\right]\beta
$$

$$
\left[(\alpha+\frac{1}{\sigma})(1-\tau_k)^2\right]\beta^2-\left[(\alpha+\frac{1}{\sigma}+(1-\alpha)\tau_k)(1-\tau_k)\rho\right]\beta-\left[\frac{1}{\alpha}((1-\alpha)^2+(\alpha+\frac{1}{\sigma})(1-\alpha))(1-\tau_k)\rho^2\right]
$$

$$
\phi,\lambda=\frac{\left[(\alpha+\frac{1}{\sigma}+(1-\alpha)\tau_k)(1-\tau_k)\rho\right]\pm\left(\begin{matrix}\left[(\alpha+\frac{1}{\sigma}+(1-\alpha)\tau_k)(1-\tau_k)\rho\right]^2\\+4\left[(\alpha+\frac{1}{\sigma})(1-\tau_k)^2\right]\left[\frac{1}{\alpha}((1-\alpha)^2+(\alpha+\frac{1}{\sigma})(1-\alpha))(1-\tau_k)\rho^2\right]\end{matrix}\right.}{2\left[(\alpha+\frac{1}{\sigma})(1-\tau_k)^2\right]}
$$

There are three eigenvalues, or roots to this equation, which we call ϕ, λ, β. The expressions for them are:

$$
\begin{aligned}
\phi &= \frac{\left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)(1-\tau_k)\rho\right] + \Bigg(\begin{array}{c} \left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)(1-\tau_k)\rho\right]^2 \\ +4\left[(\alpha + \frac{1}{\sigma})(1-\tau_k)^2\right]\left[\frac{1}{\alpha}((1-\alpha)^2 + (\alpha + \frac{1}{\sigma})(1-\alpha))(1-\tau_k)\rho^2\right] \end{array}}{2\left[(\alpha + \frac{1}{\sigma})(1-\tau_k)^2\right]} \\
\lambda &= \frac{\left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)(1-\tau_k)\rho\right] - \Bigg(\begin{array}{c} \left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)(1-\tau_k)\rho\right]^2 \\ +4\left[(\alpha + \frac{1}{\sigma})(1-\tau_k)^2\right]\left[\frac{1}{\alpha}((1-\alpha)^2 + (\alpha + \frac{1}{\sigma})(1-\alpha))(1-\tau_k)\rho^2\right] \end{array}}{2\left[(\alpha + \frac{1}{\sigma})(1-\tau_k)^2\right]} \\
\beta &= 0
\end{aligned}
$$

These can be simplified a bit:

$$
\begin{aligned}
\phi &= \frac{\left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)\rho\right] + \Big(\left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)\rho\right]^2 + 4(\alpha + \frac{1}{\sigma})\left[\frac{1}{\alpha}((1-\alpha)^2 + (\alpha + \frac{1}{\sigma})(1-\alpha))(1-\tau_k)\right.}{2\left[(\alpha + \frac{1}{\sigma})(1-\tau_k)\right]} \\
\lambda &= \frac{\left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)\rho\right] - \Big(\left[(\alpha + \frac{1}{\sigma} + (1-\alpha)\tau_k)\rho\right]^2 + 4(\alpha + \frac{1}{\sigma})\left[\frac{1}{\alpha}((1-\alpha)^2 + (\alpha + \frac{1}{\sigma})(1-\alpha))(1-\tau_k)\right.}{2\left[(\alpha + \frac{1}{\sigma})(1-\tau_k)\right]} \\
\beta &= 0
\end{aligned}
$$

The first terms in the numerators of ϕ and λ are positive, as are the denominators. The absolute values of the second terms in the numerator are necessarily larger than the first terms', so ϕ is positive and λ is negative. The eigenvectors of the matrix A that correspond to these three eigenvalues can be derived for a given set of parameter values with standard mathematical software. Call the matrix of eigenvectors V. Then, we can describe the paths of the log values of c, n, k with

$$
\begin{aligned}
\ln c_t &= \ln c^* + v_{11}e^{\phi t}b_1 + v_{12}e^{\lambda t}b_2 + v_{13}e^{\beta t}b_3 \\
\ln n_t &= \ln n^* + v_{21}e^{\phi t}b_1 + v_{22}e^{\lambda t}b_2 + v_{23}e^{\beta t}b_3 \\
\ln k_t &= \ln k^* + v_{31}e^{\phi t}b_1 + v_{32}e^{\lambda t}b_2 + v_{33}e^{\beta t}b_3
\end{aligned}
$$

where v_{ij} is the i,jth component of the matrix of eigenvectors, and b_1, b_2, b_3 are coefficients that we must determine with boundary conditions.

For our boundary conditions, consider the case of $t \to \infty$. By assumption $\phi > 0$, but $\lim_{t\to\infty} c_t = c^*$, so we know that $b_1 = 0$. Similarly, given that $\beta = 0$, if b_3 were not equal to zero, the variables would not approach their steady state values as $t \to \infty$. Thus, $b_3 = 0$. That leaves us with:

$$
\begin{aligned}
\ln c_t &= \ln c^* + v_{12}e^{\lambda t}b_2 \\
\ln n_t &= \ln n^* + v_{22}e^{\lambda t}b_2 \\
\ln k_t &= \ln k^* + v_{32}e^{\lambda t}b_2
\end{aligned}
$$

At $t = 0$, we know that the capital stock is fixed at its initial level, k_0. We also know that $k_0 = k_\varepsilon$, so:

$$b_2 = \frac{(\ln k_0 - \ln k^*)}{v_{32}} = \frac{(\ln k_\varepsilon - \ln k^*)}{v_{32}}$$

We can substitute this in to our other conditions to rewrite our system:

$$\begin{aligned} \ln c_t - \ln c^* &= \frac{(\ln k_\varepsilon - \ln k^*)}{v_{32}} v_{12} e^{\lambda t} && (39) \\ \ln n_t - \ln n^* &= \frac{(\ln k_\varepsilon - \ln k^*)}{v_{32}} v_{22} e^{\lambda t} && (40) \\ \ln k_t - \ln k^* &= (\ln k_\varepsilon - \ln k^*)\, e^{\lambda t} && (41) \end{aligned}$$

$$\begin{aligned} \text{thus, at } t &= \varepsilon, \\ (\ln n_\varepsilon - \ln n^*)\, e^{-\lambda \varepsilon} &= \frac{(\ln k_\varepsilon - \ln k^*)}{v_{32}} v_{22}, \\ \text{so, } \ln n_t - \ln n^* &= (\ln n_\varepsilon - \ln n^*)\, e^{\lambda t - \lambda \varepsilon} \end{aligned}$$

As $\varepsilon \to 0$, this result implies that the rate of transition of the labor supply from its level instantly after the tax cut approaches λ:

$$\ln n_t - \ln n^* = (\ln n_\varepsilon - \ln n^*)\, e^{\lambda t} \tag{42}$$

Thus, results (41) and (42) imply that the rate of transition from the immediately post-tax cut levels of k and n to their post-tax cut steady-state levels is the rate λ.

As discussed briefly above, the nature of our system is that, when taxes are changed, consumption and labor supply are free to jump immediately, while the capital stock is momentarily fixed at its original level. We want to derive the values of c, n, k at three points in time, $t = 0, \varepsilon$, and the long run (steady state). We will use our steady state conditions to derive the pre-tax cut levels c_0, n_0, and k_0 and the steady-state values c^*, n^*, k^*. Because $k_0 = k_\varepsilon$,we can then plug the steady state values of consumption, labor supply, and capital and the initial level of capital into this system and calculate c_ε and n_ε, the jump values of consumption and labor supply, at $t = \varepsilon$. The convergence of c and n to their steady state levels begins at the values to which they jump, c_ε and n_ε, and is at the rate λ.

To find c_0, n_0, and k_0 and the steady-state values c^*, n^*, k^*, the steady state conditions we need are:

$$
\begin{aligned}
y &= k^{\alpha} n^{1-\alpha} \\
r &= \alpha k^{\alpha-1} n^{1-\alpha} \\
w &= (1-\alpha) k^{\alpha} n^{-\alpha} \\
n &= \left(\frac{(1-\tau_n) w}{\theta(1+\frac{1}{\sigma}) c} \right)^{\frac{1}{\frac{1}{\sigma}}} \\
r &= \frac{\rho}{1-\tau_k} \\
c &= y
\end{aligned}
$$

Note that we have continued to set $g = 0$ and $\gamma = 1$ for simplicity. These yield

$$
c^* = \left(\frac{\rho}{\alpha(1-\tau_k)} \right)^{\frac{\alpha}{\alpha-1}} \left(\frac{(1-\tau_n)(1-\alpha)}{\theta(1+\frac{1}{\sigma})} \right)^{\frac{1}{1+\frac{1}{\sigma}}} \tag{43}
$$

$$
n^* = \left(\frac{(1-\tau_n)(1-\alpha)}{\theta(1+\frac{1}{\sigma})} \right)^{\frac{1}{1+\frac{1}{\sigma}}} \tag{44}
$$

$$
k^* = \left(\frac{\rho}{\alpha(1-\tau_k)} \right)^{\frac{1}{\alpha-1}} \left(\frac{(1-\tau_n)(1-\alpha)}{\theta(1+\frac{1}{\sigma})} \right)^{\frac{1}{1+\frac{1}{\sigma}}} \tag{45}
$$

$$
c_0 = \left(\frac{\rho}{\alpha(1-\tau_{k,0})} \right)^{\frac{\alpha}{\alpha-1}} \left(\frac{(1-\tau_{n,0})(1-\alpha)}{\theta(1+\frac{1}{\sigma})} \right)^{\frac{1}{1+\frac{1}{\sigma}}} \tag{46}
$$

$$
n_0 = \left(\frac{(1-\tau_{n,0})(1-\alpha)}{\theta(1+\frac{1}{\sigma})} \right)^{\frac{1}{1+\frac{1}{\sigma}}} \tag{47}
$$

$$
k_0 = k_\varepsilon = \left(\frac{\rho}{\alpha(1-\tau_{k,0})} \right)^{\frac{1}{\alpha-1}} \left(\frac{(1-\tau_{n,0})(1-\alpha)}{\theta(1+\frac{1}{\sigma})} \right)^{\frac{1}{1+\frac{1}{\sigma}}} \tag{48}
$$

Note that equations (43)-(45) and (46)-(48) differ only in the tax rates that apply: τ_k, τ_n or $\tau_{k,0}, \tau_{n,0}$,where the latter are pre-tax cut, the former are post-tax cut.

Now, we plug these values into (40)-(41).

$$
\begin{aligned}
\ln n_\varepsilon - \ln n^* &= \frac{(\ln k_\varepsilon - \ln k^*)}{v_{32}} v_{22} e^{\lambda \varepsilon} \\
\ln k_\varepsilon - \ln k^* &= (\ln k_\varepsilon - \ln k^*) \, e^{\lambda \varepsilon}
\end{aligned}
$$

As $\varepsilon \to 0$, these simplify to:

$$
\ln n_\varepsilon = \ln n^* + \frac{(\ln k_\varepsilon - \ln k^*)}{v_{32}} v_{22} \tag{49}
$$

$$
\ln k_\varepsilon = \ln k_0 \tag{50}
$$

giving us our "jump" values of n and k.

With our values for n, k before the tax cut from (47) and (48), immediately after the tax cut from (49) and (50), and in the steady state from equations (44) and (45), we can calculate the present value of the transition path of tax revenues. Specifically, let R_0, R_ε, and R^* denote the pre-tax cut, jump level, and steady state tax revenues per period. We can write

$$\begin{aligned} R_0 &= \left(\alpha\tau_k + (1-\alpha)\tau_n\right) k_0^\alpha n_0^{1-\alpha} \\ R_\varepsilon &= \left(\alpha\tau_k + (1-\alpha)\tau_n\right) k_\varepsilon^\alpha n_\varepsilon^{1-\alpha} \\ R^* &= \left(\alpha\tau_k + (1-\alpha)\tau_n\right) {k^*}^\alpha {n^*}^{1-\alpha} \end{aligned}$$

11.0.3 General labor disutility function is equivalent to specific

$$H = e_t^{-\rho t}\frac{\left(c_t e^{gt}\right)^{1-\gamma} e^{(1-\gamma)v(N)} - 1}{1-\gamma} + \varphi(t)\left[(1-\tau_n)wN + (1-\tau_k)rk - c - gk + T\right]$$

Performing the household's maximization, we can derive the following results:

$$\begin{aligned} FOC_c &: \quad e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c(t)^{-\gamma} = \varphi \\ \frac{dFOC_c}{dt} &: \quad \dot{\varphi} = \left[\begin{array}{c} -\rho e^{-\rho t}e^{gt(1-\gamma)}e^{(1-\gamma)v(N)}c^{-\gamma} + \\ (1-\gamma)ge^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma} + \\ (1-\gamma)v'(N)\dot{N}e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma} - \\ \gamma e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma-1}\dot{c} \end{array}\right] \quad (14) \\ -((1-\tau_k)r - g)e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma} &= \left[\begin{array}{c} -\rho e^{-\rho t}e^{gt(1-\gamma)}e^{(1-\gamma)v(N)}c^{-\gamma} + \\ (1-\gamma)ge^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma} + \\ (1-\gamma)v'(N)\dot{N}e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma} - \\ \gamma e^{-\rho t}e^{(1-\gamma)gt}e^{(1-\gamma)v(N)}c^{-\gamma-1}\dot{c} \end{array}\right] \\ -((1-\tau_k)r - g) &= -\rho + (1-\gamma)g + (1-\gamma)v'(N)\dot{N} - \gamma c^{-1}\dot{c} \\ -(1-\tau_k)r + g &= -\rho + (1-\gamma)g + (1-\gamma)v'(N)N\frac{\dot{N}}{N} - \gamma\frac{\dot{c}}{c} \\ (1-\tau_k)r &= \rho + \gamma g + \gamma\frac{\dot{c}}{c} + (1-\gamma)v'(N)N\frac{\dot{N}}{N} \\ \frac{\dot{c}}{c} &= \frac{1}{\gamma}\left[(1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha} - \rho - \gamma g - (1-\gamma)v'(N)N\frac{\dot{N}}{N}\right] \quad (15) \end{aligned}$$

$$
\begin{array}{rcl}
FOC_N & : & e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N) = -(1-\tau_n)w(t)\varphi(t) \\
\frac{dFOC_N}{dt} & : & \left(\begin{array}{c} -\rho e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)+ \\ (1-\gamma)ge^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)+ \\ (1-\gamma)\dot{c}c^{-1}e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)+ \\ (1-\gamma)v'(N)e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)\dot{N}+ \\ v''(N)\dot{N}e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)} \end{array}\right) \\
& = & -(1-\tau_n)[\dot{w}(t)\varphi(t)+w(t)\dot{\varphi}(t)] \\
& : & \left(\begin{array}{c} -\rho e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)+ \\ (1-\gamma)ge^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)+ \\ (1-\gamma)\dot{c}c^{-1}e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)+ \\ (1-\gamma)v'(N)e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)\dot{N}+ \\ v''(N)\dot{N}e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)} \end{array}\right) \\
& = & \left\{-(1-\tau_n)\left[\begin{array}{c} \left((1-\alpha)\alpha\left(\frac{\dot{k}}{k}-\frac{\dot{N}}{N}\right)k^{\alpha}N^{-\alpha}\right)\frac{e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)}{-(1-\tau_n)((1-\alpha)k^{\alpha}N^{-\alpha})} \\ -((1-\alpha)k^{\alpha}N^{-\alpha})\left((1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha}-g\right)\frac{e^{-\rho t}e^{gt(1-\gamma)}c(t)^{1-\gamma}e^{(1-\gamma)v(N)}v'(N)}{-(1-\tau_n)((1-\alpha)k^{\alpha}N^{-\alpha})} \end{array}\right]\right\}
\end{array}
$$

$$
\begin{array}{rcl}
& : & -\rho+(1-\gamma)g+\frac{(1-\gamma)}{\gamma}\left[(1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha}-\rho-\gamma g-(1-\gamma)v'(N)N\frac{\dot{N}}{N}\right]+(1-\gamma)v'(N)\dot{N}+\frac{v''}{v} \\
& = & \alpha\left(\frac{\dot{k}}{k}-\frac{\dot{N}}{N}\right)-\left((1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha}-g\right) \\
& : & \frac{\dot{N}}{N}\left[\alpha-\frac{(1-\gamma)}{\gamma}(1-\gamma)v'(N)N+(1-\gamma)v'(N)N+\frac{v''(N)\dot{N}}{v'(N)}\right] \\
& = & \rho-(1-\gamma)g-\frac{(1-\gamma)}{\gamma}\left[(1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha}-\rho-\gamma g\right]+\alpha\left[k^{\alpha-1}N^{1-\alpha}-\frac{c}{k}-g\right]-\left((1-\tau_k)\alpha k^{\alpha-1}N\right. \\
\frac{\dot{N}}{N} & = & \frac{\rho-(1-\gamma)g-\frac{(1-\gamma)}{\gamma}\left[(1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha}-\rho-\gamma g\right]+\alpha\left[k^{\alpha-1}N^{1-\alpha}-\frac{c}{k}-g\right]-\left((1-\tau_k)\alpha k^{\alpha-1}N^{1-}\right.}{\left[\alpha-\frac{(1-\gamma)}{\gamma}(1-\gamma)v'(N)N+(1-\gamma)v'(N)N+\frac{v''(N)\dot{N}}{v'(N)}\right]} \\
\text{if } \gamma & = & 1\text{, simplify to } \frac{\dot{N}}{N}=\frac{\rho+\tau_k\alpha k^{\alpha-1}N^{1-\alpha}+(1-\alpha)g-\alpha\frac{c}{k}}{\alpha+\frac{v''(N)\dot{N}}{v'(N)}}
\end{array}
$$

Now,

$$\frac{v'(N)}{v''(N)\cdot N}=\sigma$$

So,

$$\frac{\dot{N}}{N}=\frac{\rho+\tau_k\alpha k^{\alpha-1}N^{1-\alpha}+(1-\alpha)g-\alpha\frac{c}{k}}{\alpha+\frac{1}{\sigma}}$$

$$
\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\gamma}\left[(1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha} - \rho - \gamma g - (1-\gamma)v'(N)N\frac{\dot{N}}{N}\right] && (16)\\
\text{if } \gamma &= 1, && (17)\\
\frac{\dot{c}_t}{c_t} &= (1-\tau_k)\alpha k^{\alpha-1}N^{1-\alpha} - \rho - g && (18)\\
\frac{\dot{N}}{N} &= \frac{\rho + \tau_k \alpha k^{\alpha-1}N^{1-\alpha} + (1-\alpha)g - \alpha\frac{c}{k}}{\alpha + \frac{1}{\sigma}} && (34)
\end{aligned}
$$

These are identical to when the specific functional form was assumed.